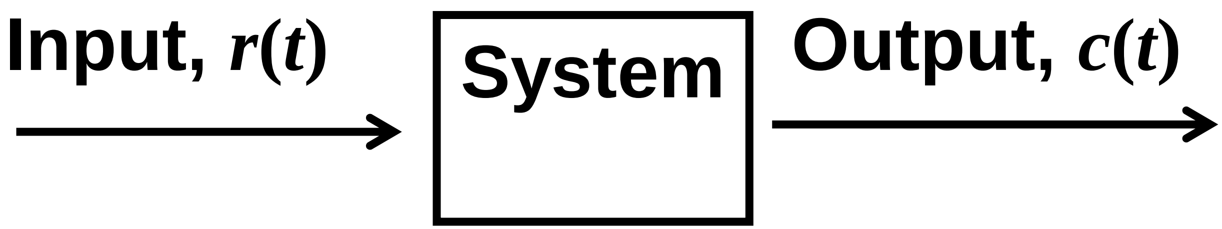
# Transfer Function

- **Introduce a new concept: Transfer Function**




**It's governing equation:**

$$A_n \frac{d^n c(t)}{dt^n} + \cdots + A_1 \frac{dc(t)}{dt} + A_0 c(t) = B_m \frac{d^m r(t)}{dt^m} + \cdots + B_1 \frac{dr(t)}{dt} + B_0 r(t)$$

# Transfer Function

- **Laplace Transfer ( assuming zero initial condition)**

$$\left(A_n s^n + \cdots + A_1 s + A_0\right)C(s) = \left(B_m s^m + \cdots + B_1 s + B_0\right)R(s)$$

- **Rearrange:**

$$\frac{C(s)}{R(s)} = G(s) = \frac{B_m s^m + \cdots + B_1 s + B_0}{A_n s^n + \cdots + A_1 s + A_0}$$

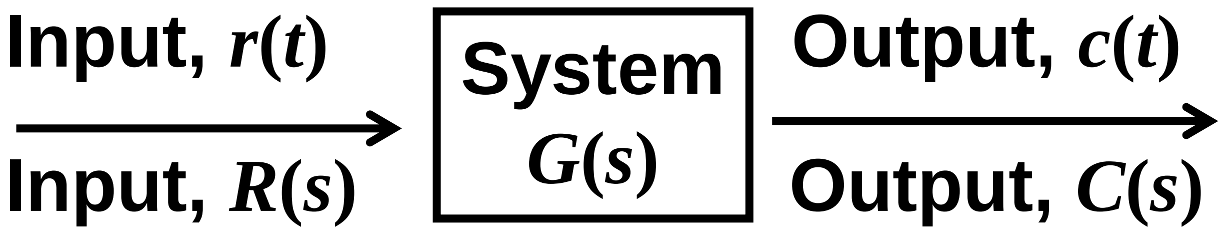
**Definition of the Transfer Function!**

# Transfer Function

**The Transfer function $G(s)$ is a property of the system elements only, and is not dependent on the excitation and initial conditions. In addition, transfer functions can be used to represent both closed-loop and open-loop systems**

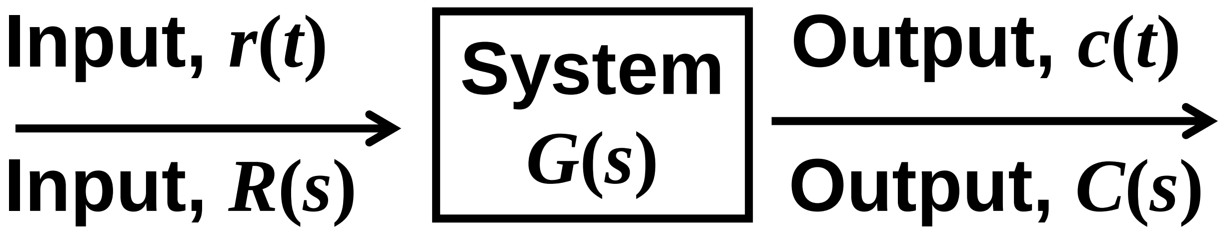
# Transfer Function

- **Block diagram**



# Transfer Function

- **Block diagram**

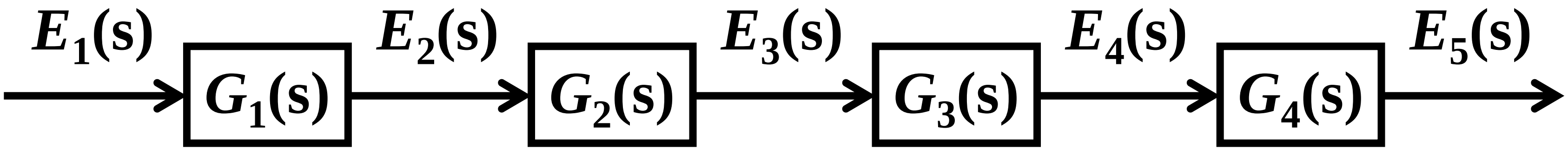



- **Laplace Transform of the output**

$$C(s) = G(s)R(s)$$

# Transfer Function of Systems

- **Cascaded system**




$$E_2(s) = G_1(s)E_1(s)$$

$$E_3(s) = G_2(s)E_2(s)$$

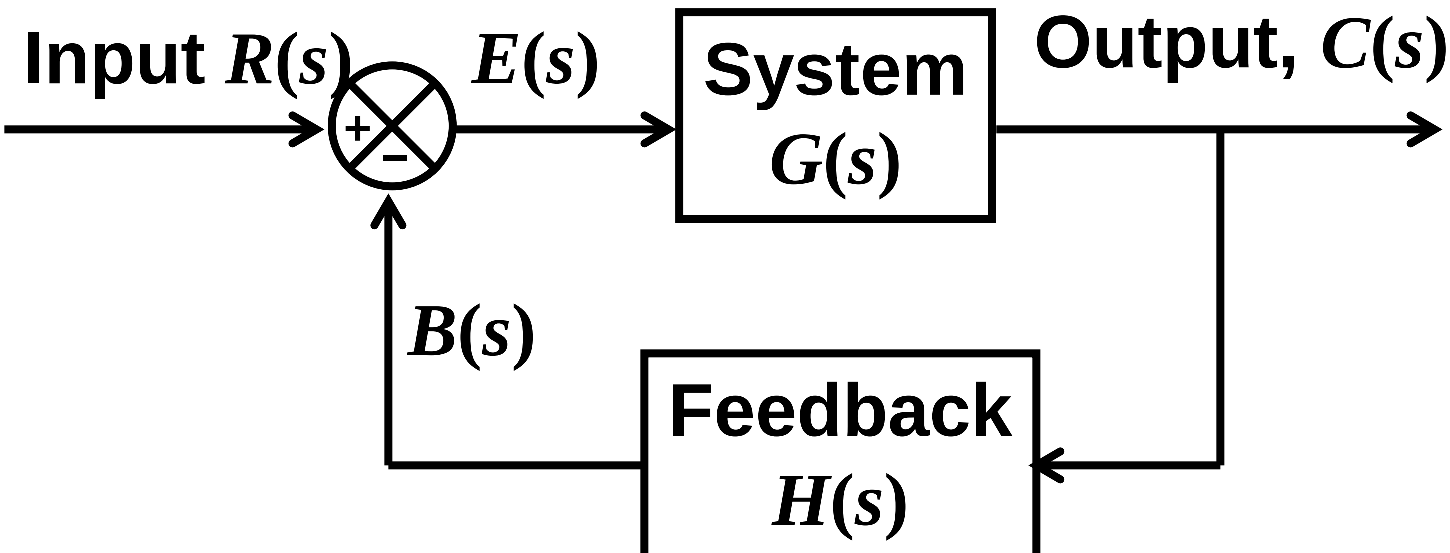
$$E_4(s) = G_3(s)E_3(s)$$

$$E_5(s) = G_4(s)E_4(s)$$

$$E_5 / E_1 = G_1(s)G_2(s)G_3(s)G_4(s)$$

# Transfer Function of Systems

- Cascaded system
- Single-loop feedback system



# Transfer Function of Single-loop Feedback System

- **Use the definition of transfer function:**

$$\begin{cases} B(s) = H(s)C(s) \\ E(s) = R(s) - B(s) \\ C(s) = G(s)E(s) \end{cases}$$

- **Solve for $C(s)/R(s)$**

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

# Transfer Function of Single-loop Feedback System

$$\frac{C(s)}{R(s)} = \left.\frac{G(s)}{1+G(s)H(s)}\right|_{G(s)H(s)>>1} \approx \frac{1}{H(s)}$$

**Independent of *G*(*s*) !**

# Transfer Function of Single-loop feedback system

- **Characteristic equation (denominator)**

$$1+G(s)H(s)=0$$

- **Solving for error**

$$\frac{C(s)=G(s)E(s)}{R(s)}=\frac{G(s)}{1+G(s)H(s)}$$

# Transfer Function of Single-loop feedback system

**Therefore**

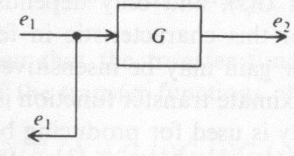
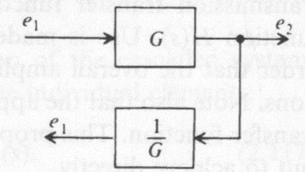
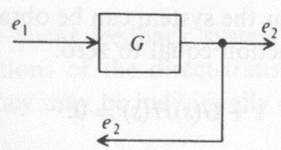
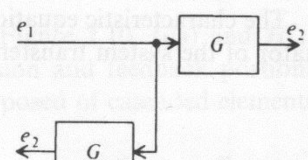
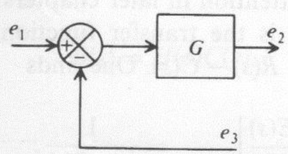
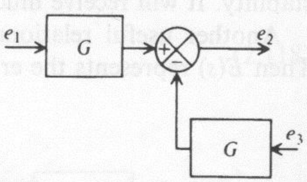
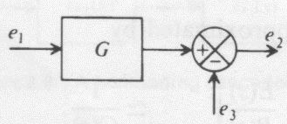
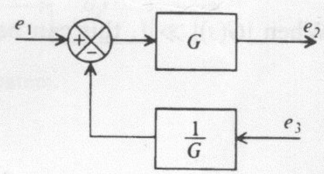
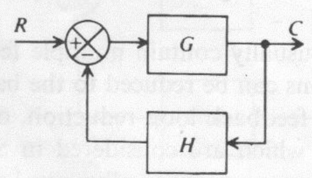
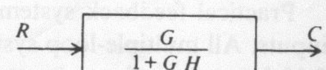
$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)H(s)}$$

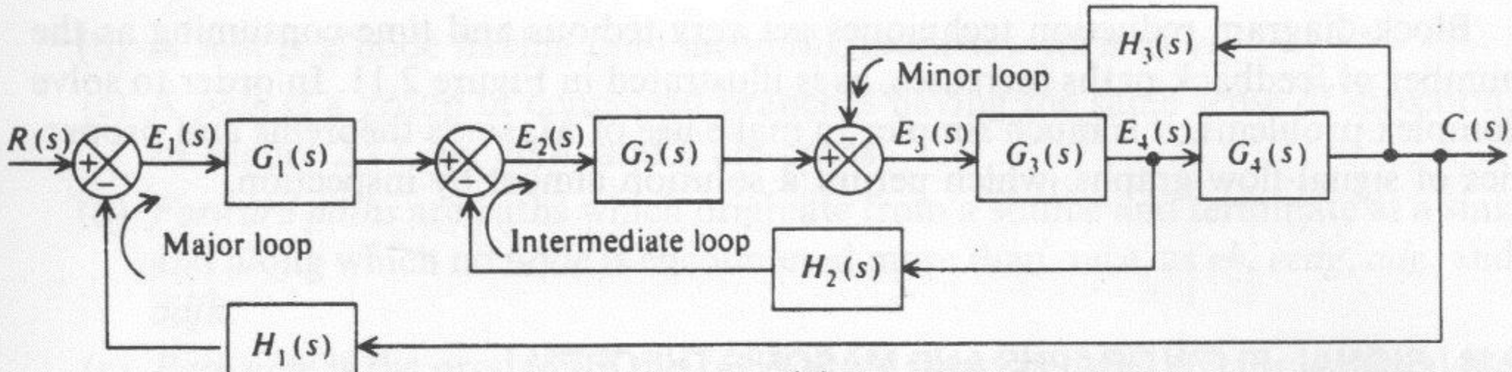
$$\left.\frac{E(s)}{R(s)}\right|_{H(s)=1} = \left.\frac{1}{1+G(s)}\right|_{G(s)>>1} \approx \frac{1}{G(s)}$$

**Error should be very small, if *G*(s) is large !**

# Block diagram transformation

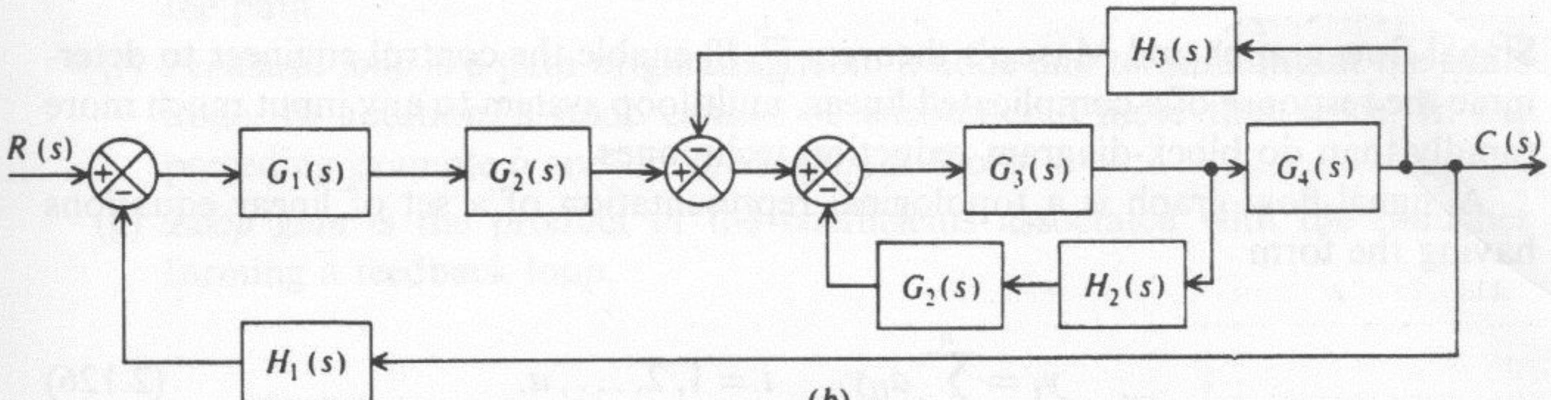
Table 2.5. Block Diagram Transformations

| Transformation | Original block diagram | Equivalent block diagram |
|---|---|---|
| 1. Moving a pickoff point behind a block |  |  |
| 2. Moving a pickoff point ahead of a block |  |  |
| 3. Moving a summing point behind a block |  |  |
| 4. Moving a summing point ahead of a block |  |  |
| 5. Eliminating a feedback loop |  |  |



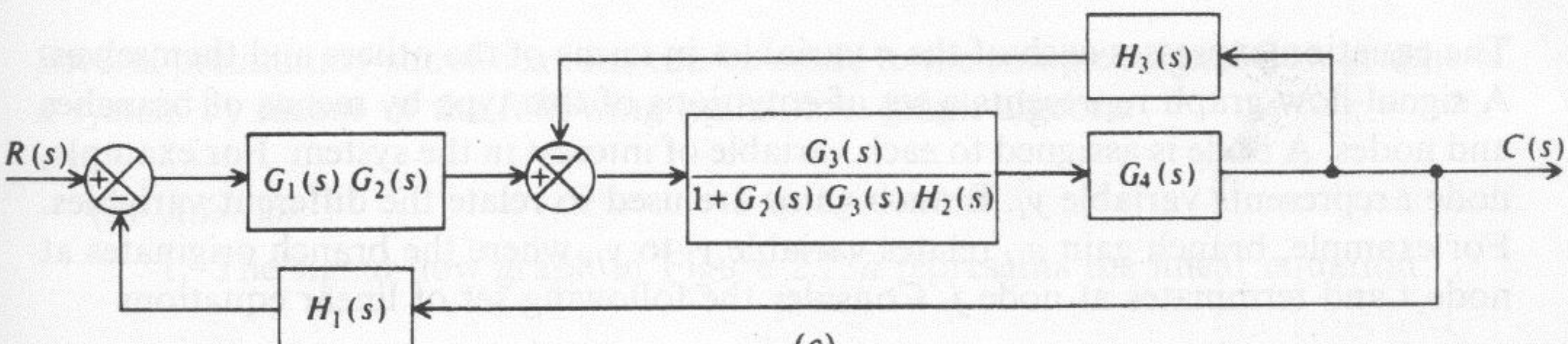
R(s)
E1(s)
G1(s)
E2(s)
G2(s)
E3(s)
G3(s)
E4(s)
G4(s)
C(s)
H3(s)
Minor loop
Major loop
Intermediate loop
H2(s)
H1(s)


(a)



R(s)
G1(s)
G2(s)
H3(s)
G3(s)
G4(s)
C(s)
G2(s)
H2(s)
H1(s)


(b)



R(s)
G1(s) G2(s)
H3(s)
G3(s)
1+G2(s) G3(s) H2(s)
G4(s)
C(s)
H1(s)


(c)

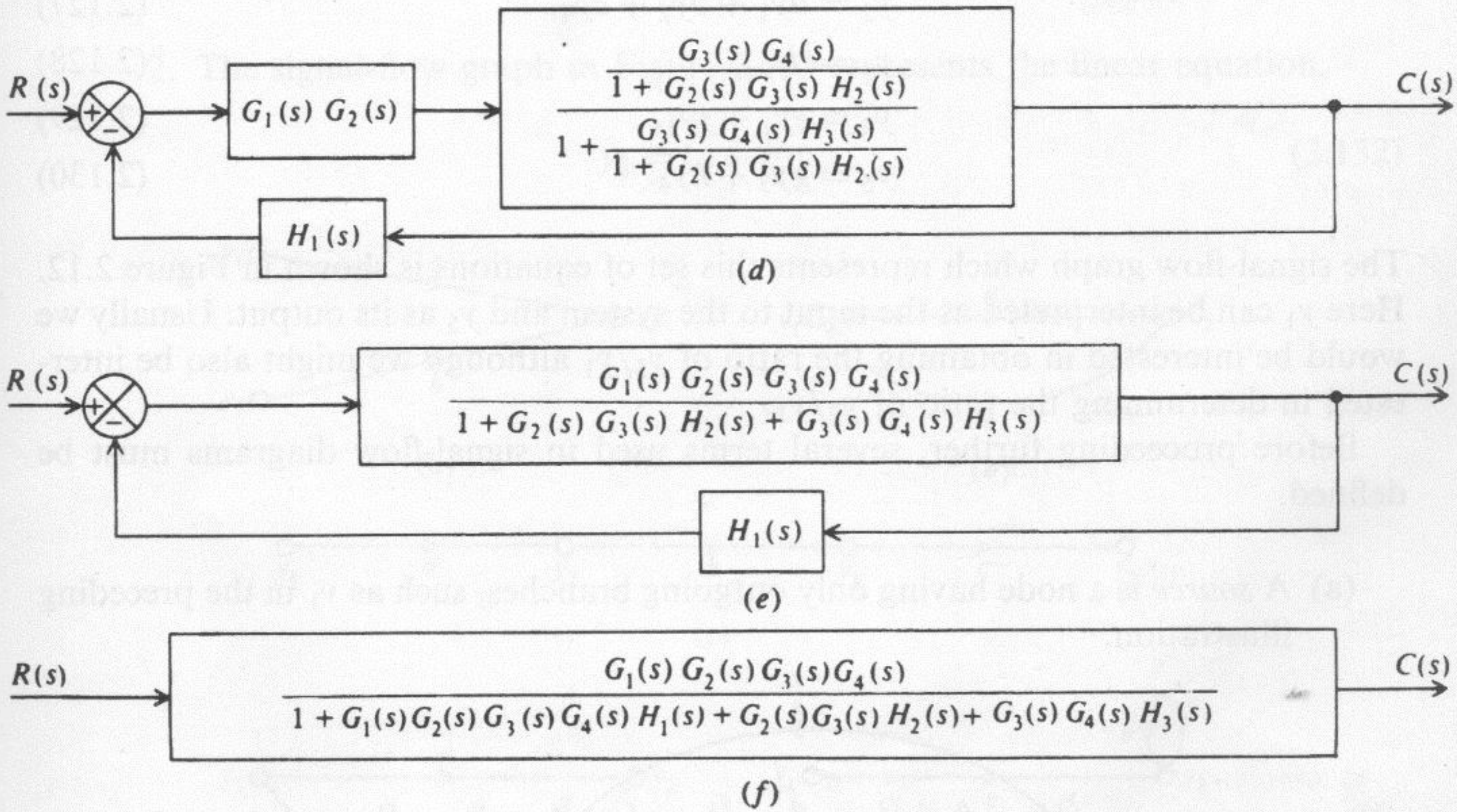
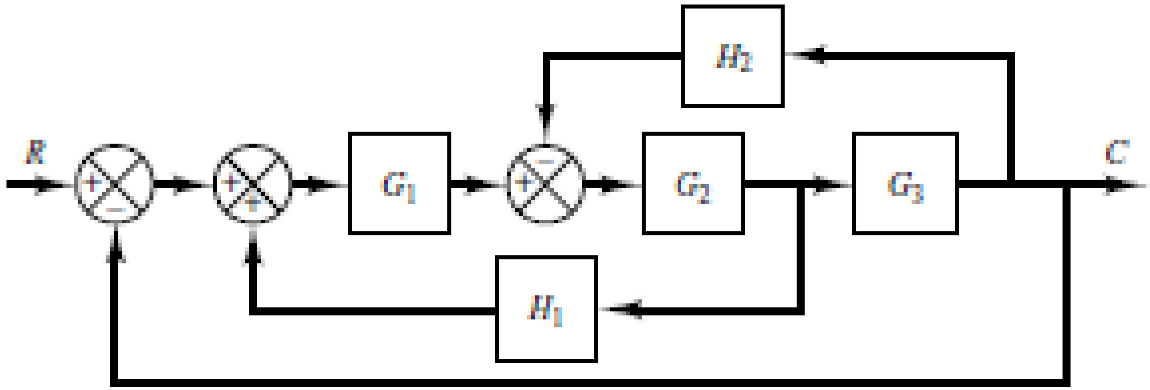



**Figure 2.11** Reducing a multiple-loop system containing complex paths. (*a*) The original system. (*b*) Rearrangement of the summing points of the intermediate and minor loops. (*c*) Reduction of the equivalent intermediate loop. (*d*) Reduction of the equivalent minor loop. (*e*) The equivalent feedback system. (*f*) The system transfer function.

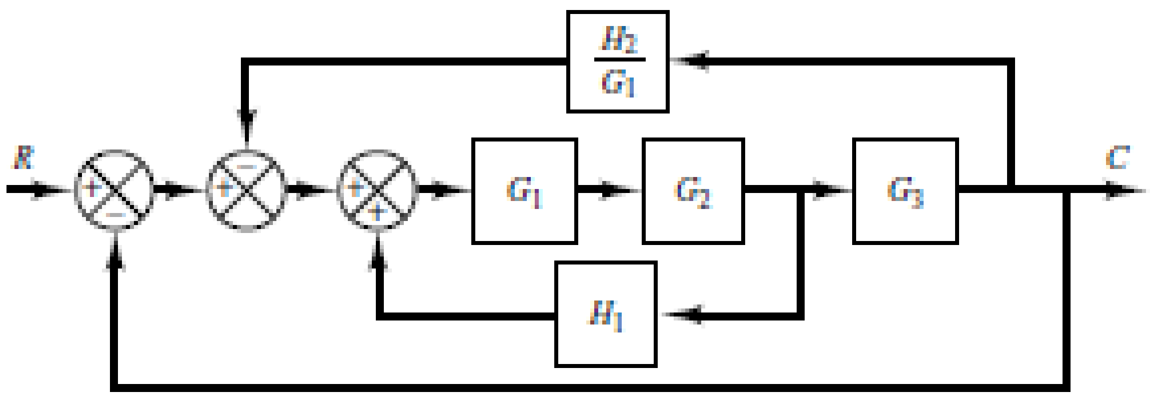
**EXAMPLE 2–1**

- **Consider the system shown in Figure 2–13(a). Simplify this diagram.**
- **By moving the summing point of the negative feedback loop containing H2 outside the positive**
- **feedback loop containing H1, we obtain Figure 2–13(b). Eliminating the positive feedback loop,**
- **we have Figure 2–13(c).The elimination of the loop containing H2/G1 gives Figure 2–13(d).Finally,**
- **eliminating the feedback loop results in Figure 2–13(e).**

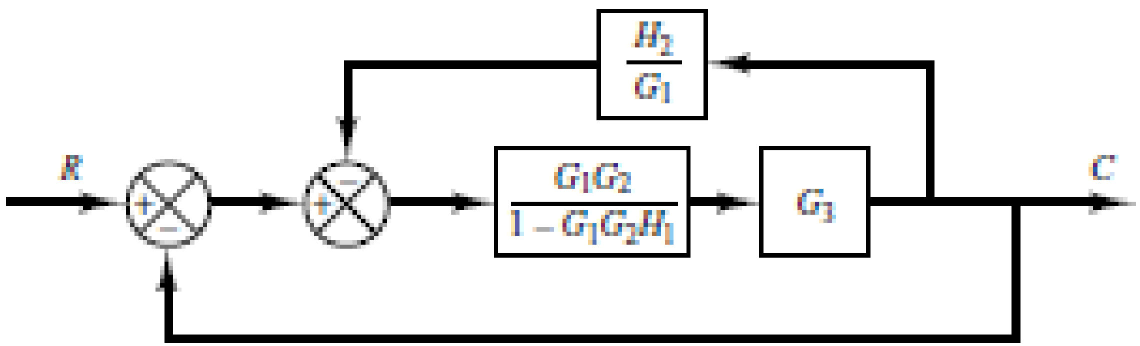
(a)




(b)

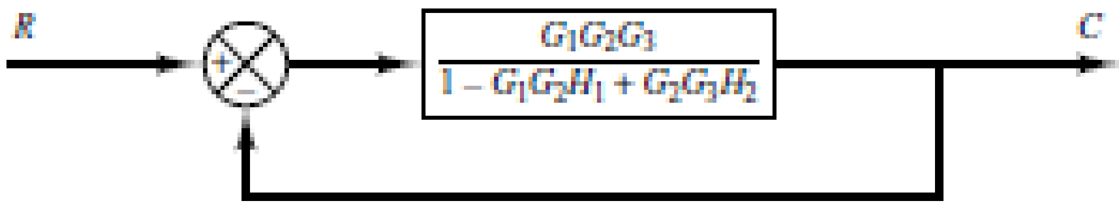


(c)



$R$
$\frac{H_2}{G_1}$
$\frac{G_1G_2}{1 - G_1G_2H_1}$
$G_3$
$C$


(d)

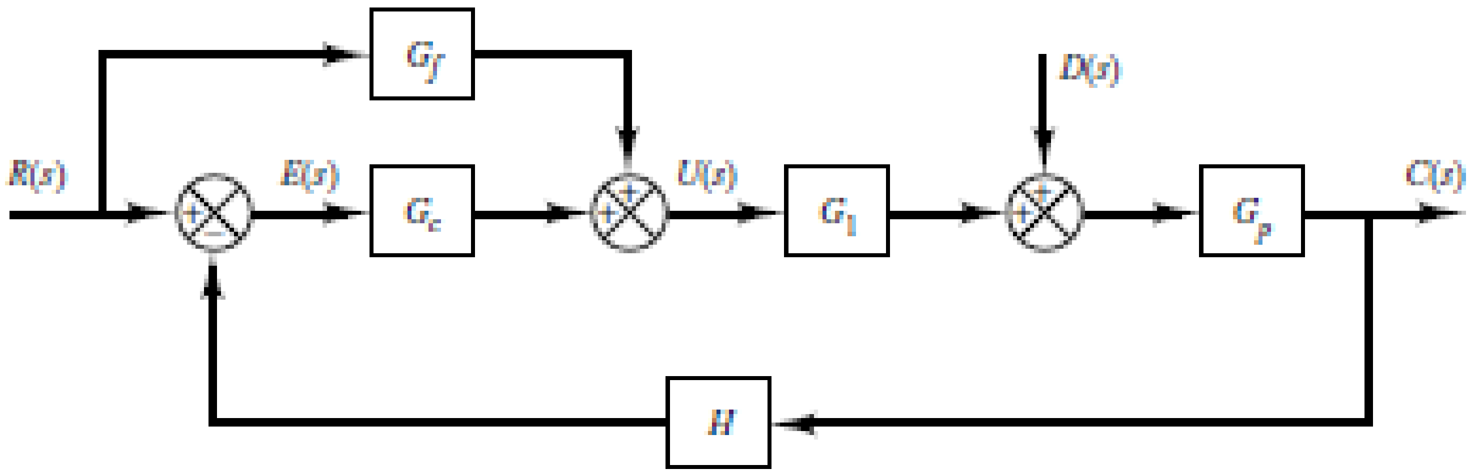


$R$
$\frac{G_1G_2G_3}{1 - G_1G_2H_1 + G_2G_3H_2}$
$C$


(e)

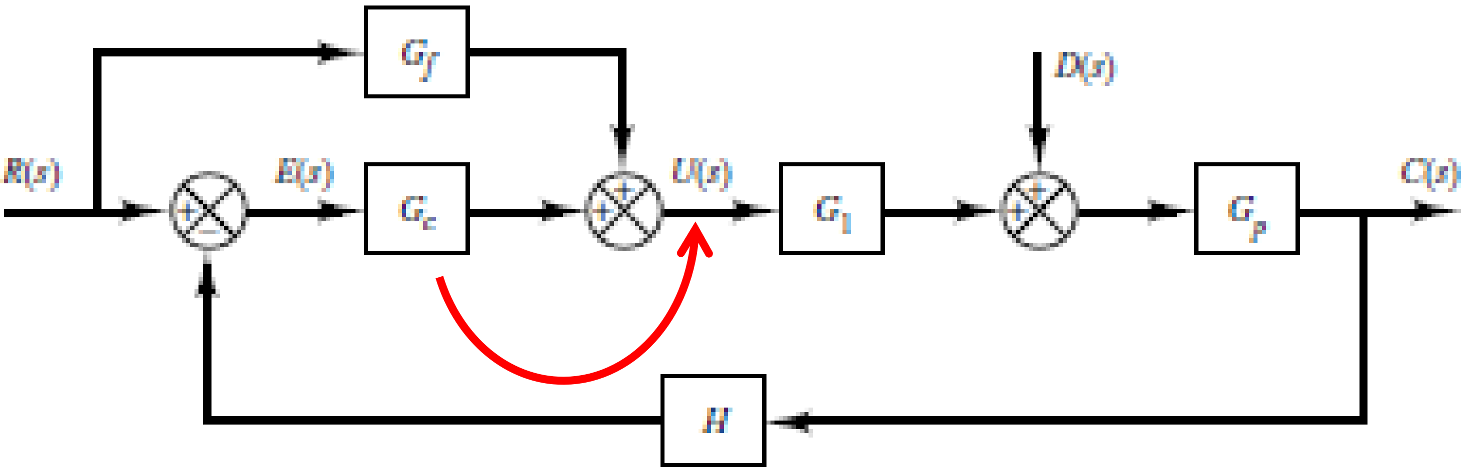
$R$
$\frac{G_1G_2G_3}{1 - G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3}$
$C$

# Quiz III 15min

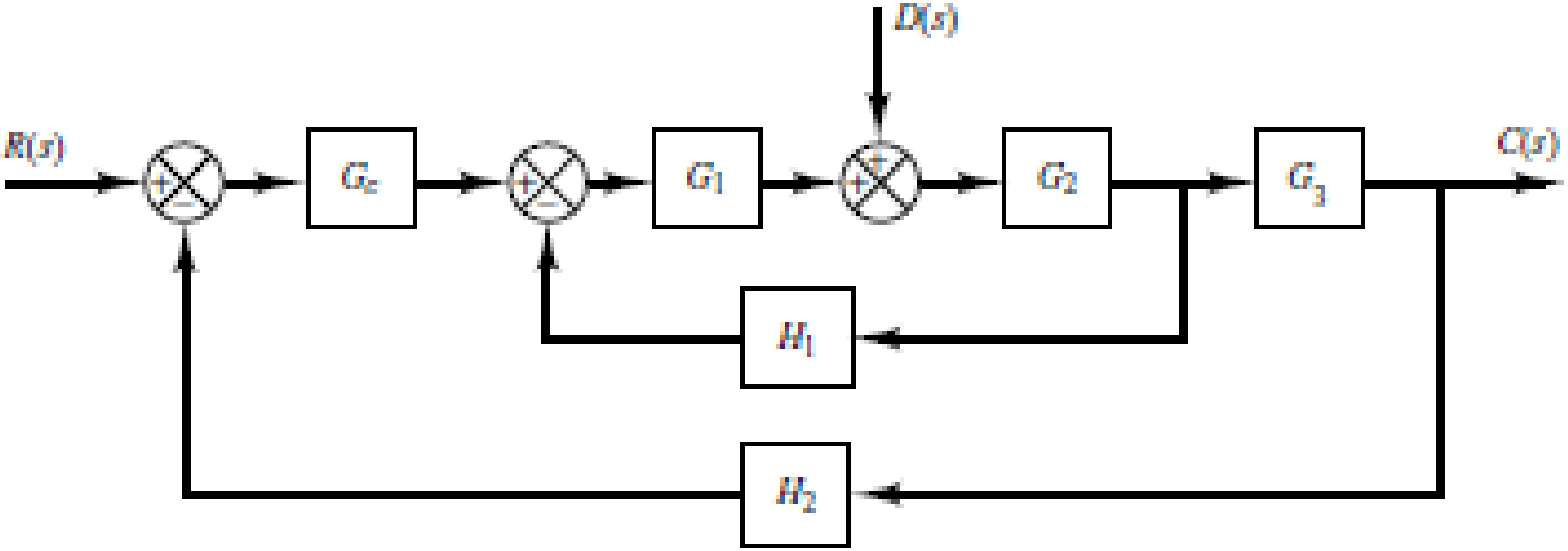



$$\frac{C(s)}{R(s)} = \frac{G_1G_p(G_f + G_c)}{1 + G_1G_pG_cH} \qquad \frac{C(s)}{D(s)} = \frac{G_p}{1 + G_1G_pG_cH}$$

# Answer



# Homework II



# Signal Flow Graphs



and

$$Z_b = \frac{8.00(s+1.5)}{s^2+2s+5} + \frac{12.00(s+4)}{s^2+4s+13} + \frac{4.47337}{s+4}.$$

Deriving the element values from the above, we finally obtain the lattice shown in Fig. 6. This lattice has the desired transfer impedance.

### Conclusion

A simple method has been demonstrated for the realization of any minimum-phase or nonminimum-phase transfer impedance as an open-circuited lattice. The arms of the lattice are of a simple form and contain no mutual inductance. Any inductance used in the lattice always appears with an associated series resistance so that low-$Q$ coils may be used in building the network. The procedure presented allows a measure of control over the $Q$'s of the coils used in the final network.

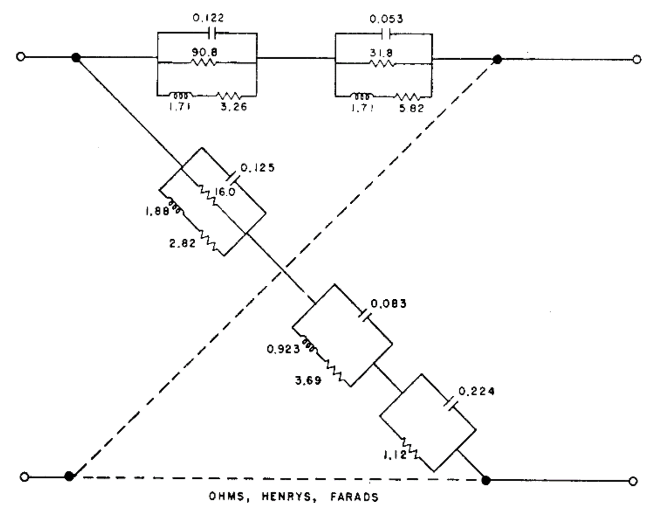



Fig. 6—Lattice obtained for illustrative example where $Z_{12}=p/q$.

# FEEDBACK THEORY—Some Properties of Signal Flow Graphs*

SAMUEL J. MASON†, SENIOR MEMBER, IRE

# Signal Flow Graphs



Conclusion

The elliptic function transformation (1). is used here for the purpose of locating zeros and poles of a low-pass filter network function. Charts of the type shown in Figs. 7 to 12 may be prepared for any range of application whenever desired. The compactness of the expressions that give the tolerance and other characteristic quantities makes the preparation of these charts which represent a whole group of network functions with many singularities a matter of evaluating only a few terms together with a few rational operations. These charts, after they are prepared, will be very helpful for design purposes. For instance, if a required attenuation beyond twice the cut-off frequency must be greater than 13 db, Fig. 10 indicates that a filter function with the charge arrangement of Fig. 3(b) and values of $a$ and $c$ of 0.810 and 0.673 respectively will satisfy the requirement. The locations of all poles and zeros of this filter are determined in the $z$-plane. The locations of zeros and poles in the $s$-plane may be found by applying the inverse transformation.

## Feedback Theory—Further Properties of Signal Flow Graphs*

SAMUEL J. MASON†

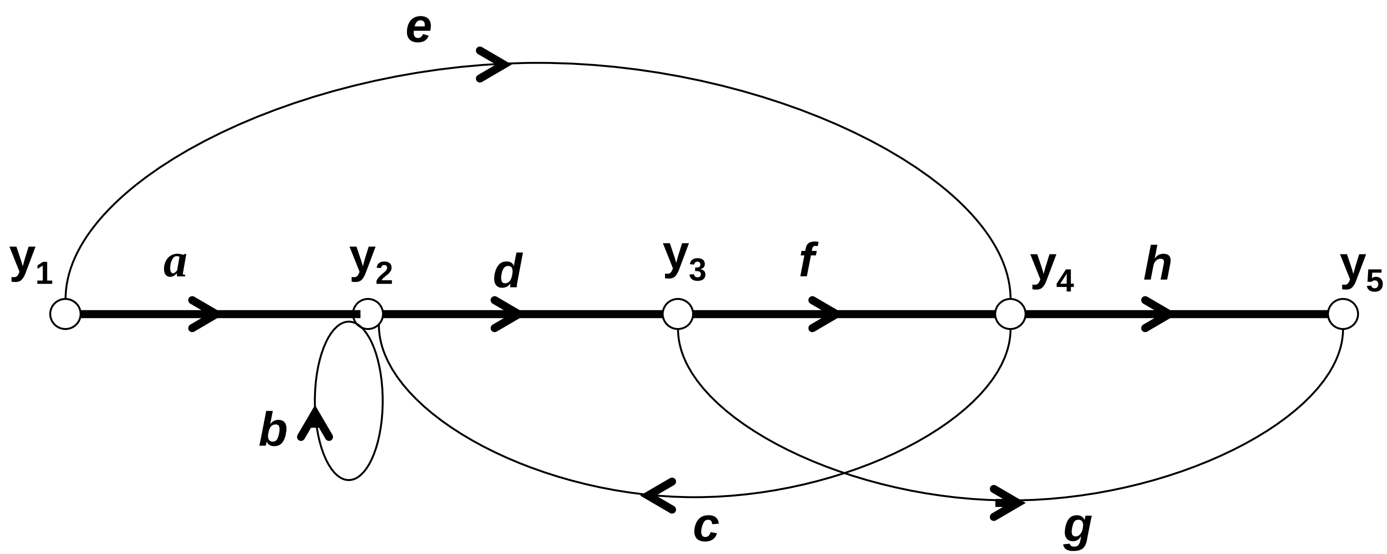
# Signal-Flow Graphs and Mason's Theorem

**A signal-flow graph is a topological representation of a set of linear equations having**

$$y_i = \sum_{j}^{n} a_{ij} y_j \qquad i = 1, 2, \cdots, n$$

$$\begin{cases} y_2 = a y_1 + b y_2 + c y_4 \\ y_3 = d y_2 \\ y_4 = e y_1 + f y_3 \\ y_5 = g y_3 + h y_4 \end{cases}$$

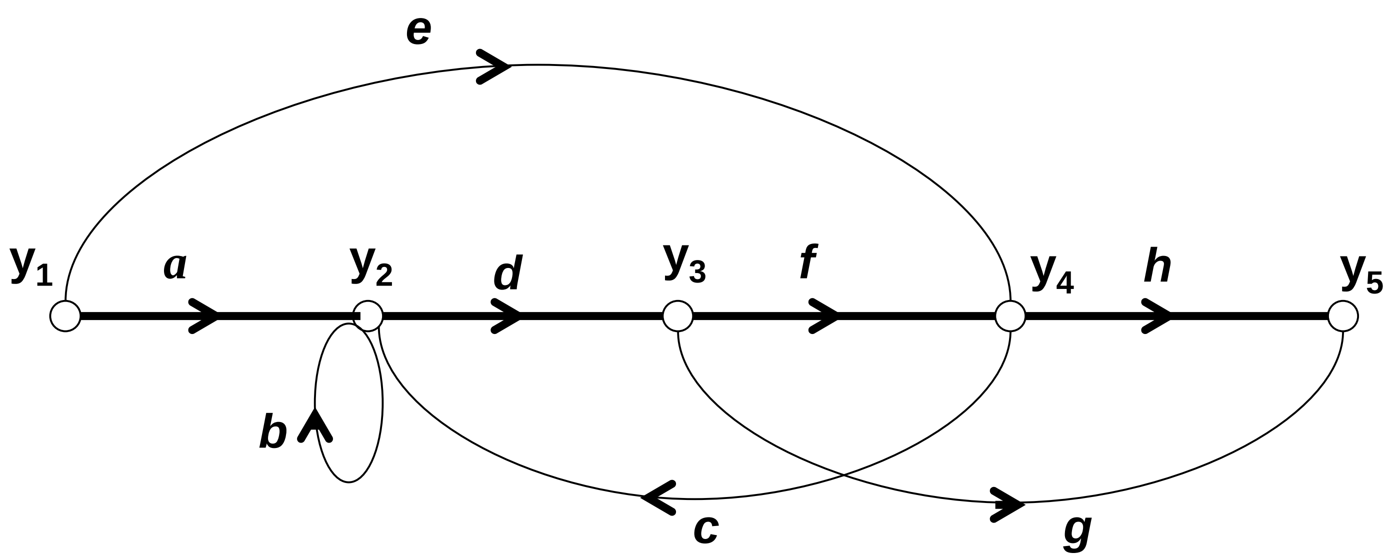
# Signal-Flow Graph




– A ***source*** is node having only outgoing branches $y_1$

– A ***sink*** is a node having only incoming branches $y_5$

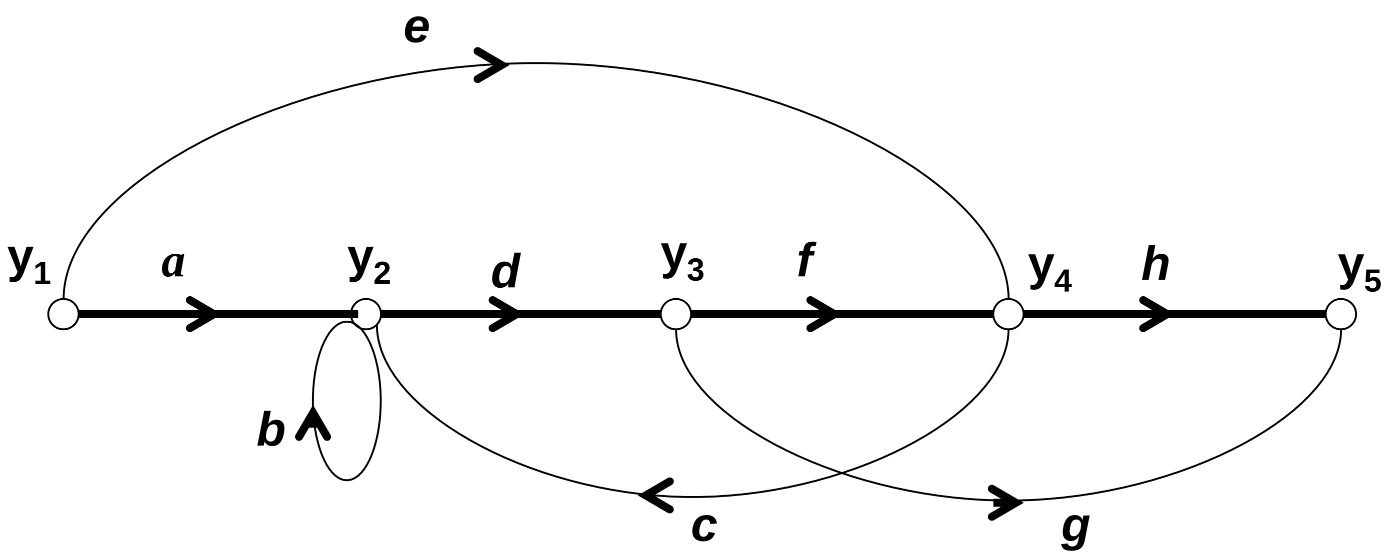
# Signal-Flow Graph




– A ***path*** is a group of connected branches having the same sense of direction (***eh b***)

– ***Forward paths*** are paths which originate from a source and terminate at a sink and along which no node is encountered more than once (***eh adfh b***)

# Signal-Flow Graph

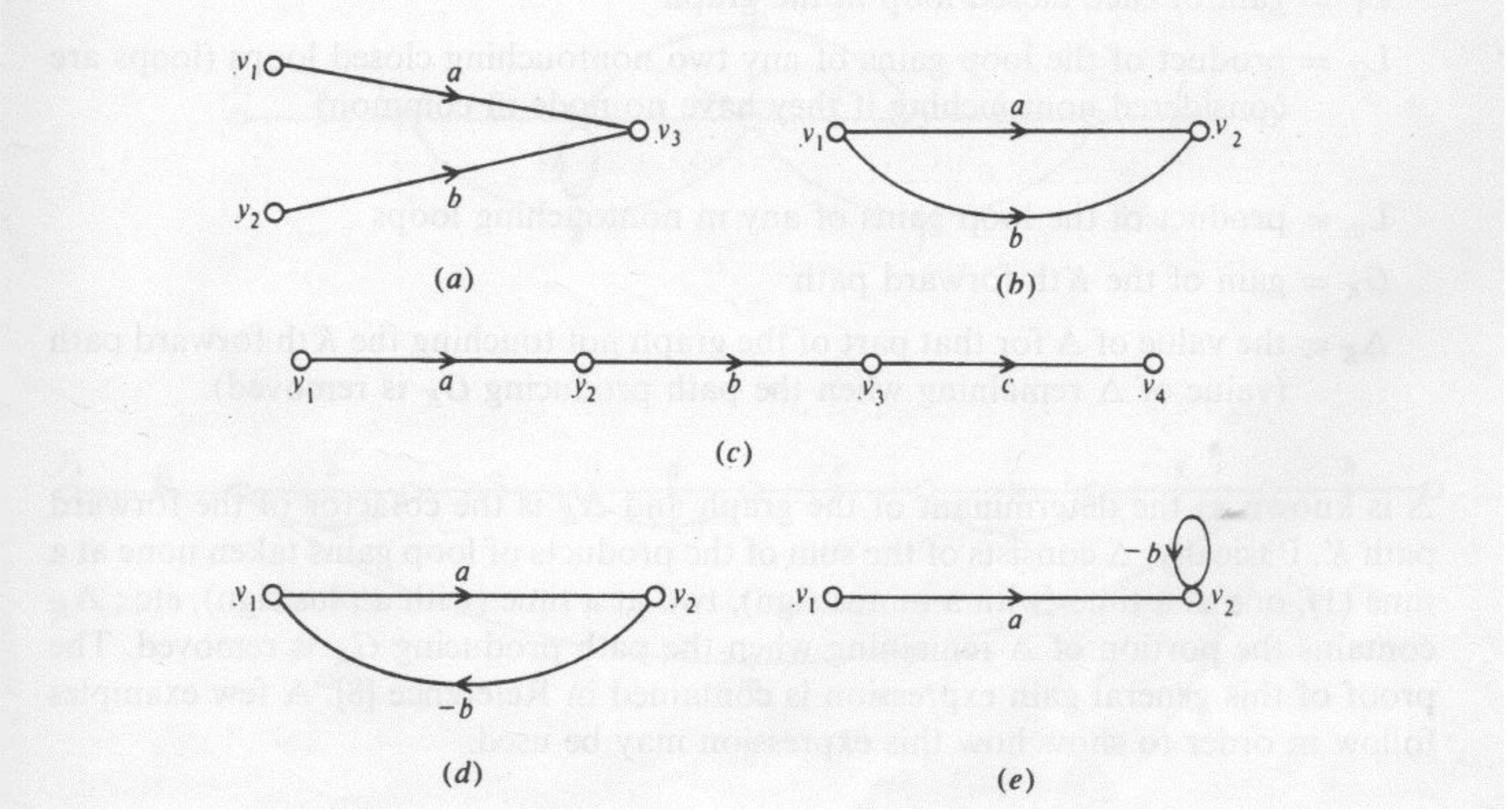



– ***Feedback loop*** **is a path originating from a node and terminating at the same node. In addition, a node cannot be encountered more than once (*b dfc*)**

# Signal-Flow Graph

- ***Path gain*** **is the product of the coefficients associated with the branches along the path**
- ***Loop gain*** **is the product of the coefficients associated with the branches forming a feed back loop**

# Reduction of the signal-flow-graph



(a) *Addition*

1. The signal-flow graph in Figure 2.13*a* represents the linear equation

$$y_3 = ay_1 + by_2. \tag{2.131}$$

2. The signal-flow graph in Figure 2.13*b* represents the linear equation

$$y_2 = (a + b)y_1. \tag{2.132}$$

(b) *Multiplication.* The signal-flow graph in Figure 2.13*c* represents the linear equation

$$y_4 = abcy_1. \tag{2.133}$$

(c) *Feedback loops*

1. The signal-flow graph in Figure 2.13*d* represents the linear equation

$$y_2 = \frac{a}{1 + ab}y_1. \tag{2.134a}$$

2. The signal-flow graph in Figure 2.13*e* represents the linear equation

$$y_2 = \frac{a}{1 + b}y_1. \tag{2.134b}$$

# Reduction of the Signal-Flow Graph

- **Signal-Flow Graph Reduction**
  - **Addition**
  - **Multiplication**
  - **Feedback loops**
- **Mason's theorem**

$$G = \frac{\sum_k G_k \Delta_k}{\Delta}$$

where

$$\Delta = 1 - \sum L_1 + \sum L_2 - \sum L_3 + \cdots + (-1)^m \sum L_m$$

$L_1$ = gain of each closed loop in the graph

$L_2$ = product of the loop gains of any two nontouching closed loops (loops are considered nontouching if they have no node in common)

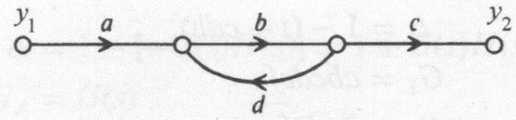
. . . . . . . . . . . . . . . . . . . . .

$L_m$ = product of the loop gains of any m nontouching loops

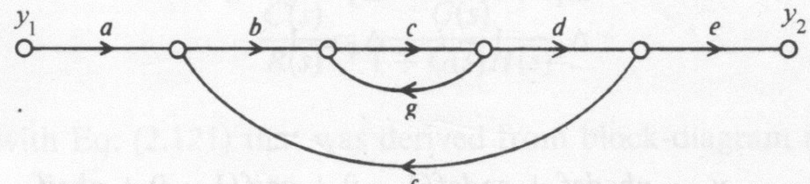
$G_K$ = gain of the $K$th forward path

$\Delta_K$ = the value of $\Delta$ for that part of the graph not touching the $K$th forward path (value of $\Delta$ remaining when the path producing $G_K$ is removed).

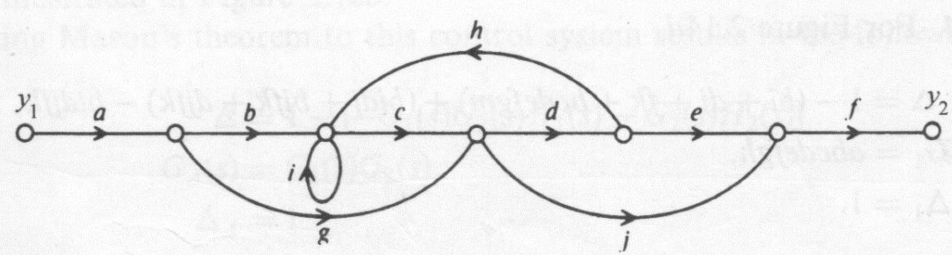
$\Delta$ is known as the determinant of the graph and $\Delta_K$ is the cofactor of the forward path $K$. Basically, $\Delta$ consists of the sum of the products of loop gains taken none at a time (1), one at a time (with a minus sign), two at a time (with a plus sign), etc.; $\Delta_K$ contains the portion of $\Delta$ remaining when the path producing $G_K$ is removed. The proof of this general gain expression is contained in Reference [8]. A few examples follow in order to show how this expression may be used.



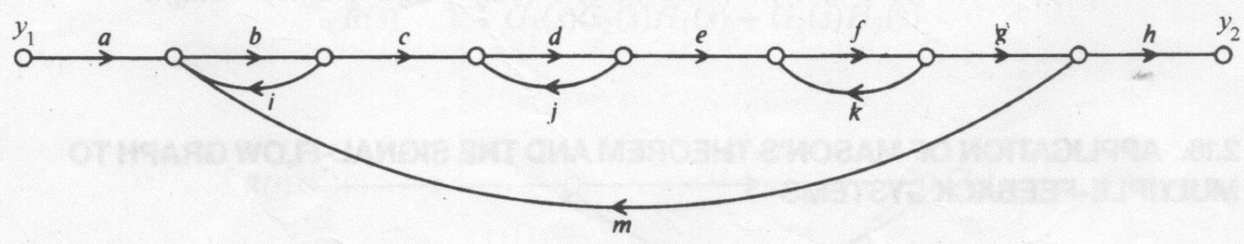

(a)




(b)




(c)




(d)

**Example 1.** For Figure 2.14*a*,

$$\Delta = 1 - bd,$$
$$G_1 = abc,$$
$$\Delta_1 = 1.$$

Therefore,

$$G = \frac{y_2}{y_1} = \frac{abc}{1 - bd}.$$

**Example 2.** For Figure 2.14*b*,

$$\Delta = 1 - cg - bcdf$$
$$G_1 = abcde.$$

Therefore,

$$\Delta_1 = 1,$$
$$G = \frac{y_2}{y_1} = \frac{abcde}{1 - cg - bcdf}.$$

**Example 3.** For Figure 2.14*c*,

$$\begin{aligned}
\Delta &= 1 - (i + cdh),\\
G_1 &= abcdef,\\
G_2 &= agdef,\\
G_3 &= agjf,\\
G_4 &= abcjf,\\
\Delta_1 &= 1, \qquad \Delta_3 = 1 - i,\\
\Delta_2 &= 1 - i, \quad \Delta_4 = 1.
\end{aligned}$$

Therefore,

$$G = \frac{y_2}{y_1} = \frac{abcdef + agdef(1-i) + agjf(1-i) + abcjf}{1 - (i + cdh)}.$$

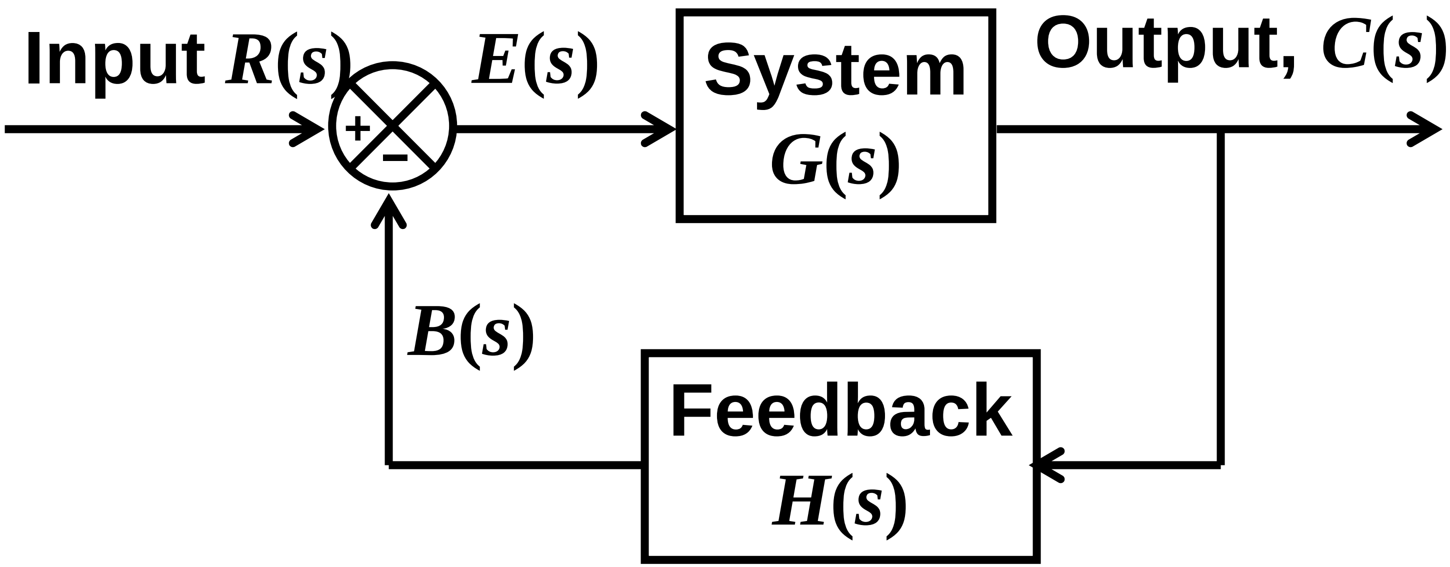
**Example 4.** For Figure 2.14*d*,

$$\begin{aligned}
\Delta &= 1 - (bi + dj + fk + bcdefgm) + (bidj + bifk + djfk) - bidjfk,\\
G_1 &= abcdefgh,\\
\Delta_1 &= 1.
\end{aligned}$$

Therefore,

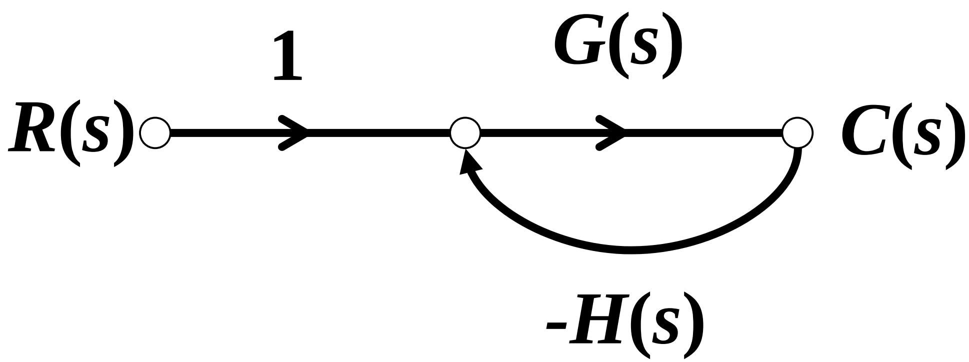
$$G = \frac{y_2}{y_1} = \frac{abcdefgh}{1 - (bi + dj + fk + bcdefgm) + (bidj + bifk + djfk) - bidjfk}.$$

# Single-loop feedback system




$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

# Apply Mason's Theorem to Single-loop feedback system



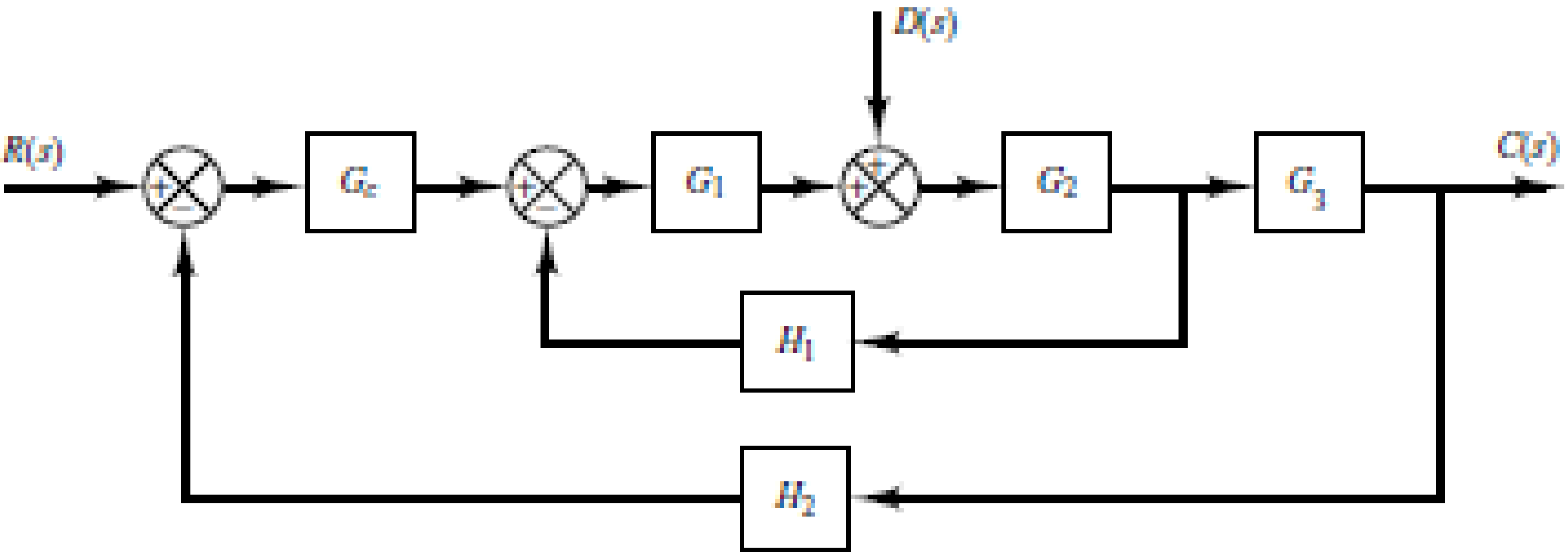

$$\Delta = 1 - \left[ -G(s)H(s) \right] = 1 + G(s)H(s)$$

$$G_A = 1 \cdot G(s) = G(s)$$

$$\Delta_A = 1$$

$$G = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

# Homework III



# Review of Matrix Algebra

- **Identity Matrix** $(a_{ii} = 1, a_{ij} = 0)$
- **Diagonal Matrix** $(a_{ii} \neq 0, a_{ij} = 0)$
- **Symmetric Matrix** $(a_{ij} = a_{ji})$
- **Skew-Symmetric Matrix**

  $(a_{ii} = 0, a_{ij} = -a_{ji})$
- **Zero Matrix**
- **Adjoint Matrix** $(a_{ij} \leftarrow A_{ij})$
- **Transpose** $(a_{ij} \leftrightarrow a_{ji})$

# Adjoint Matrix

- **Cofactor:** Cofactor $A_{ij}$.
- **The matrix B whose element in the ith row and jth column equals Aji is called the adjoint of A and is denoted by adj A, or**

$$\mathbf{B} = (b_{ij}) = (A_{ji}) = \operatorname{adj} \mathbf{A}$$

- **That is, the adjoint of A is the transpose of the matrix whose elements are the cofactors of A, or**

$$\operatorname{adj} \mathbf{A} = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

- **Inverse**

$$\mathbf{A}^{-1} = \frac{\text{adj } \mathbf{A}}{|\mathbf{A}|} = \begin{bmatrix} \dfrac{A_{11}}{|\mathbf{A}|} & \dfrac{A_{21}}{|\mathbf{A}|} & \cdots & \dfrac{A_{n1}}{|\mathbf{A}|} \\ \dfrac{A_{12}}{|\mathbf{A}|} & \dfrac{A_{22}}{|\mathbf{A}|} & \cdots & \dfrac{A_{n2}}{|\mathbf{A}|} \\ \vdots & \vdots & & \vdots \\ \dfrac{A_{1n}}{|\mathbf{A}|} & \dfrac{A_{2n}}{|\mathbf{A}|} & \cdots & \dfrac{A_{nn}}{|\mathbf{A}|} \end{bmatrix}$$

- **Example**

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & -2 \\ 1 & 0 & -3 \end{bmatrix}$$

$$|\mathbf{A}| = 17 \qquad \operatorname{adj} \mathbf{A} = \begin{bmatrix} \begin{vmatrix} -1 & -2 \\ 0 & -3 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 0 & -3 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ -1 & -2 \end{vmatrix} \\ -\begin{vmatrix} 3 & -2 \\ 1 & -3 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 1 & -3 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 3 & -2 \end{vmatrix} \\ \begin{vmatrix} 3 & -1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 6 & -4 \\ 7 & -3 & 2 \\ 1 & 2 & -7 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{\operatorname{adj} \mathbf{A}}{|\mathbf{A}|} = \begin{bmatrix} \frac{3}{17} & \frac{6}{17} & -\frac{4}{17} \\ \frac{7}{17} & -\frac{3}{17} & \frac{2}{17} \\ \frac{1}{17} & \frac{2}{17} & -\frac{7}{17} \end{bmatrix}$$

# Review of Matrix Algebra

- **Addition and subtraction**
- **Multiplication by a scalar**
- **Multiplication of two matrices**
- **Inverse of a matrix**
- **Differentiation of a matrix**
- **Integration of a matrix**

# State Variable Method

Use a representation of the system dynamics that contain the system’s input-output relationship (similar to that of a transfer function) but in terms of n first-order differential equations to represent the $n^{th}$ order system

# State Variable Method

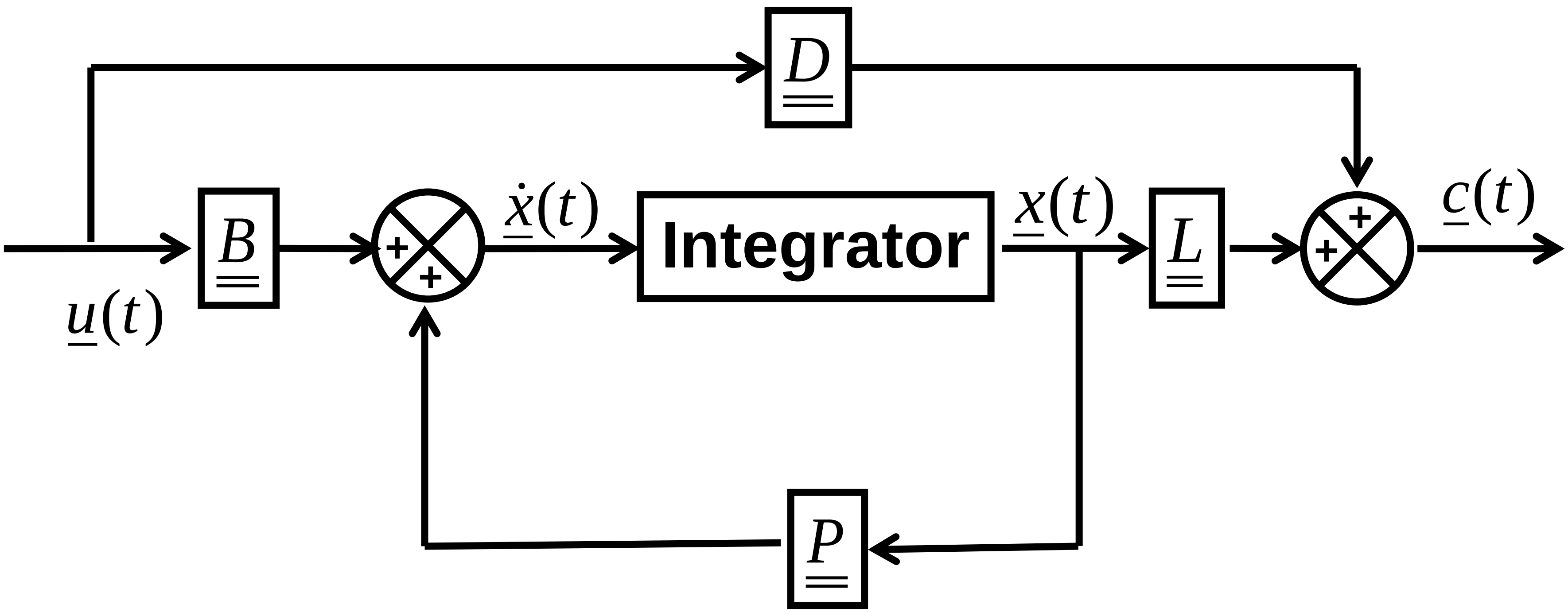
- **State Representation in Phase-Variable Canonical Form**

$$\dot{\underline{x}}(t) = \underline{\underline{P}}\,\underline{x}(t) + \underline{\underline{B}}\,\underline{u}(t)$$

**where $\underline{x}(t)$ is the state vector, $\dot{\underline{x}}(t)$ is its time derivative, $\underline{u}(t)$ is the input vector, $\underline{\underline{P}}$ is the state (companion) matrix, and $\underline{\underline{B}}$ is the input matrix**

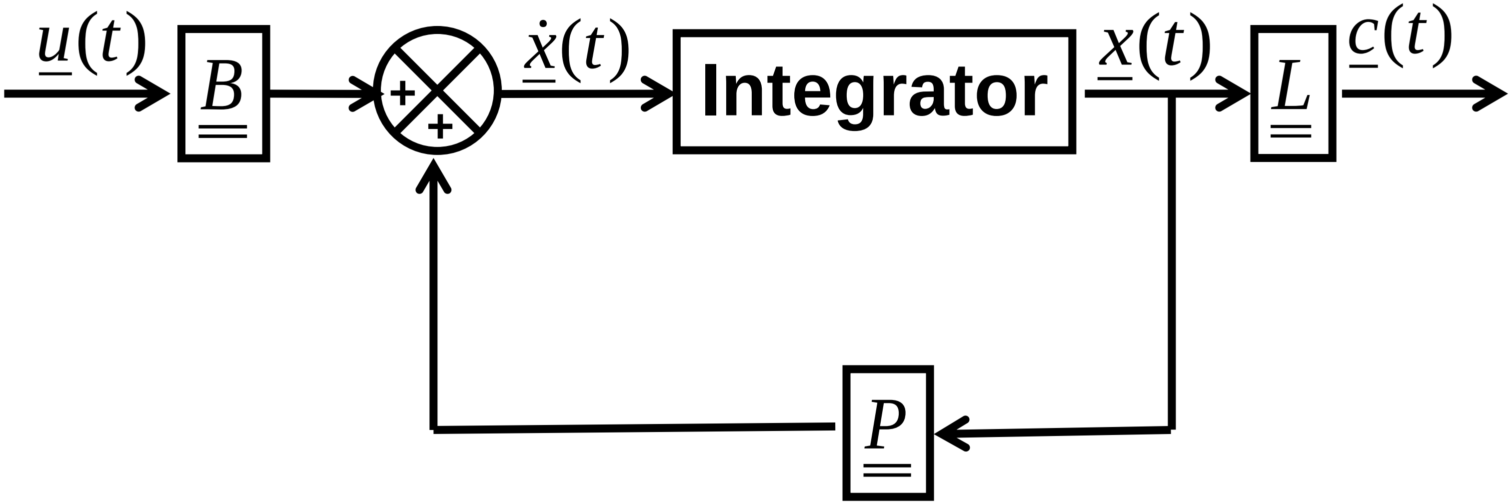
# State Variable Method

– **Block Diagram of the Phase-Variable Canonical Form (from Definition Equation)**



# State Variable Method

– **Block Diagram of the Phase-Variable Canonical Form (from Definition Equation)**



# State Variable Method

$$\underline{\underline{P}} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & & & \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}, \qquad \underline{\underline{B}} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix}$$

$$\begin{aligned} \dot{x}_1 &= p_{11}x_1(t) + \cdots + p_{1n}x_n(t) + b_{11}u_1(t) + \cdots b_{1m}u_m(t) \\ \dot{x}_2 &= p_{21}x_1(t) + \cdots + p_{2n}x_n(t) + b_{21}u_1(t) + \cdots b_{2m}u_m(t) \\ &\vdots \\ \dot{x}_n &= p_{n1}x_1(t) + \cdots + p_{nn}x_n(t) + b_{n1}u_1(t) + \cdots b_{nm}u_m(t) \end{aligned}$$

# State Variable Method

## System's Output

$$\underline{c}(t) = \underline{\underline{L}}\,\underline{x}(t) + \underline{\underline{D}}\,\underline{u}(t)$$

**where $\underline{c}(t)$ is the output vector, $\underline{\underline{L}}$ is the output matrix, $\underline{\underline{D}}$ is the coefficient matrix represents the direct transmission between input and output, in most case equal to zero. Therefore**

$$\underline{c}(t) = \underline{\underline{L}}\,\underline{x}(t)$$

# State Variable Method

- **Example**

$$P(s) = \frac{C(s)}{U(s)} = \frac{5}{s^3 + 8s^2 + 9s + 2}$$

$$\frac{d^3c(t)}{dt^3} + 8\frac{d^2c(t)}{dt^2} + 9\frac{dc(t)}{dt} + 2c(t) = 5u(t)$$

- **Define the state variable as:**

$$x_1(t) = c(t),\ x_2(t) = \dot{c}(t),\ x_3(t) = \ddot{c}(t),$$

# State Variable Method

- **We have**

$$\dot{x}_1(t) = x_2(t) = \dot{c}(t),$$
$$\dot{x}_2(t) = x_3(t) = \ddot{c}(t),$$
$$\dot{x}_3(t) = -2x_1(t) - 9x_2(t) - 8x_3(t) + 5u(t)$$

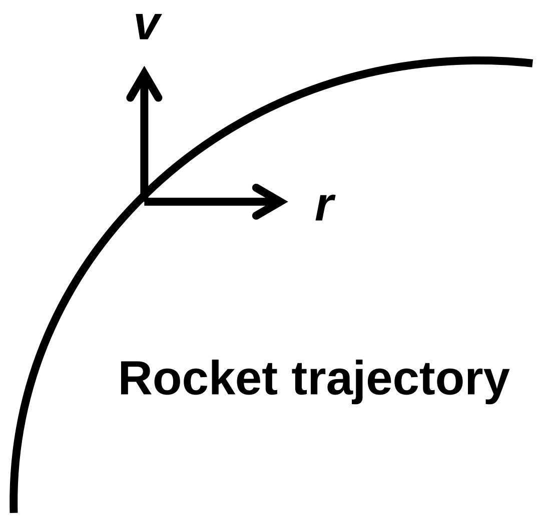
- **Recall Phase-Variable Canonical Form**

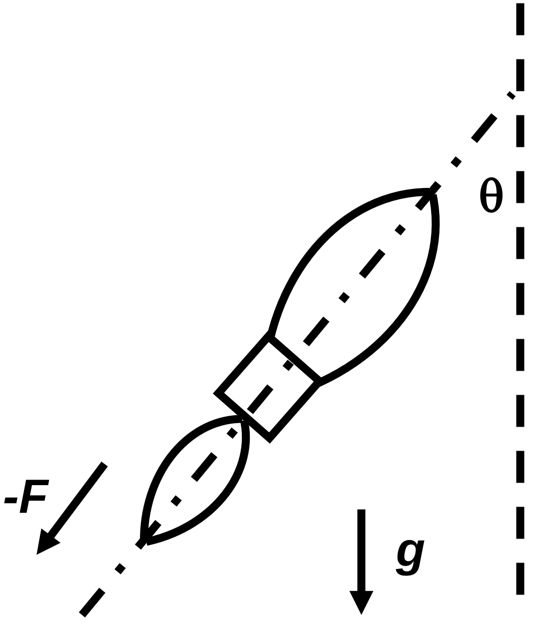
$$\dot{\underline{x}}(t) = \underline{\underline{P}}\,\underline{x}(t) + \underline{\underline{B}}\,\underline{u}(t)$$

$$\underline{c}(t) = \underline{\underline{L}}\,\underline{x}(t)$$

# State Variable Method

- **Example**



**Example 3.** In the third example used to illustrate the representation of the dynamics of a system in state-variable form, consider the problem of rocket flight in two dimensions. Representing the vertical and horizontal axes by $v(t)$ and $r(t)$, respectively, the describing equations are given by

$$\ddot{r}(t) = F(t)\cos\theta(t), \tag{2.218}$$

$$\ddot{v}(t) = F(t)\sin\theta(t) - g, \tag{2.219}$$

where $F$ is thrust force per unit mass, $\theta$ is thrust direction relative to the $r$ axis, and $g$ is the gravitational force. The control inputs are considered to be $F(t)$ and $\theta(t)$. Defining

$$\begin{aligned} x_1(t) &= r(t), & x_2(t) &= \dot{r}(t), \\ x_3(t) &= v(t), & x_4(t) &= \dot{v}(t), \\ u_1(t) &= F(t), & u_2(t) &= \theta(t), \end{aligned}$$

we find that the dynamics are described by

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= u_1(t)\cos u_2(t), \\ \dot{x}_3(t) &= x_4(t), \\ \dot{x}_4(t) &= u_1(t)\sin u_2(t) - g. \end{aligned}$$

This system can also be described in phase-variable canonical form by

$$\dot{\mathbf{x}}(t) = \mathbf{P}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \tag{2.220}$$

$$\mathbf{c}(t) = \mathbf{L}\mathbf{x}(t), \tag{2.221}$$

where

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}, \quad \dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{c}(t) = \begin{bmatrix} r(t) \\ v(t) \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{u}(t) = \begin{bmatrix} 0 \\ u_1(t)\cos u_2(t) \\ 0 \\ u_1(t)\sin u_2(t) - g \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

**2.42.** **(a)** Defining $x_1(t) = c(t)$ and $x_2(t) = \dot{c}(t)$, the plant dynamics become

$$\dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = 2x_2(t) - x_1(t),$$

or in the vector/matrix form

$$\dot{\mathbf{x}}(t) = \mathbf{P}\mathbf{x}(t), \quad \mathbf{c}(t) = \mathbf{L}\mathbf{x}(t),$$

where

$$\mathbf{x}(t) = \begin{bmatrix} x_1 t) \\ x_2(t) \end{bmatrix}, \quad \dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix},$$

$$\mathbf{L} = [1 \quad 0].$$

**(b)** With $x_1(t) = c(t)$ and $x_2(t) = \dot{c}(t)$, the plant dynamics become

$$\dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = -2x_2(t) - x_1(t) + A,$$

or in vector/matrix form

$$\dot{\mathbf{x}}(t) = \mathbf{P}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{c}(t) = \mathbf{L}\mathbf{x}(t),$$

where

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{u}(t) = \begin{bmatrix} 0 \\ A \end{bmatrix},$$

$$\mathbf{L} = [1 \quad 0].$$

(c) With $x_1(t) = c(t)$, $x_2(t) = \dot{c}(t)$, and $x_3(t) = \ddot{c}(t)$, the plant dynamics becomes

$$\dot{x}_1(t) = x_2(t),$$
$$\dot{x}_2(t) = x_3(t),$$
$$\dot{x}_3(t) = -2x_1(t) - 2x_2(t) - 3x_3(t),$$

or, in vector/matrix form,

$$\dot{\mathbf{x}}(t) = \mathbf{P}\mathbf{x}(t), \quad \mathbf{c}(t) = \mathbf{L}\mathbf{x}(t),$$

where

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}, \quad \dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -2 & -3 \end{bmatrix},$$

$$\mathbf{L} = [1 \quad 0 \quad 0].$$

**(d)** With $x_1(t) = c(t)$, $x_2(t) = \dot{c}(t)$, and $x_3(t) = \ddot{c}(t)$, the plant dynamics become

$$\dot{x}_1(t) = x_2(t),$$
$$\dot{x}_2(t) = x_3(t),$$
$$\dot{x}_3(t) = -2x_1(t) - 2x_2(t) - 3x_3(t) + A,$$

or, in vector/matrix form,

$$\dot{\mathbf{x}}(t) = \mathbf{P}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{c}(t) = \mathbf{L}\mathbf{x}(t),$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}, \quad \dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -2 & -3 \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{u}(t) = \begin{bmatrix} 0 \\ 0 \\ A \end{bmatrix}, \quad \mathbf{L} = [1 \quad 0 \quad 0].$$

# Homework IV

**2.42.** Determine the phase-variable canonical form for the systems characterized by the following differential equations:

**(a)** $$\frac{d^2c(t)}{dt^2} + 2\frac{dc(t)}{dt} + c(t) = 0,$$

**(b)** $$\frac{d^2c(t)}{dt^2} + 2\frac{dc(t)}{dt} + c(t) = A,$$

**(c)** $$\frac{d^3c(t)}{dt^3} + \frac{3d^2c(t)}{dt^2} + 2\frac{dc(t)}{dt} + 2c(t) = 0,$$

**(d)** $$\frac{d^3c(t)}{dt^3} + 3\frac{d^2c(t)}{dt^2} + 2\frac{dc(t)}{dt} + 2c(t) = A.$$

**2.43.** The approximate linear equations for a spherical satellite are given by

$$I\ddot{\theta}_1(t) + \omega_0 I\dot{\theta}_3(t) = L_1,$$
$$I\ddot{\theta}_2(t) = L_2,$$
$$I\ddot{\theta}_3(t) - \omega_0 I\dot{\theta}_1(t) = L_3,$$

where $\theta_1(t)$, $\theta_2(t)$, $\theta_3(t)$ represent angular deviaitons of the satellite from a set of axes with fixed orientation, $L_1$, $L_2$, $L_3$ represent applied torques, $I$ represents the moment of inertia, and $\omega_0$ represents the angular frequency of the oriented axis. Determine the phase-variable canonical form of the system's dynamics.

# State-Variable Diagram

## – Example

$$P(s) = \frac{C(s)}{U(s)} = \frac{s^2 + 4s + 1}{s^3 + 9s^2 + 8s}$$

**Dividing numerator and denominator by** $s^3$

$$P(s) = \frac{C(s)}{U(s)} = \frac{s^{-1} + 4s^{-2} + s^{-3}}{1 + 9s^{-1} + 8s^{-2}}$$

**Force terms in the numerator, pure integrators !**

# State-Variable Diagram

**Define the error node of the system**

$$E(s)=\frac{U(s)}{1+9s^{-1}+8s^{-2}}$$

**then**

**And** $$C(s)=\left(s^{-1}+4s^{-2}+s^{-3}\right)E(s)$$

$$E(s)=U(s)-9s^{-1}E(s)-8s^{-2}E(s)$$

**Draw diagram**

**More example ?**

# State Transition Matrix

- **Recall phase-variable canonical equation**

$$\dot{\underline{x}}(t) = \underline{\underline{P}}\,\underline{x}(t) + \underline{\underline{B}}\,\underline{u}(t)$$

- **Laplace transfer**

$$s\,\underline{X}(s) - \underline{x}(0^+) = \underline{\underline{P}}\,\underline{X}(s) + \underline{\underline{B}}\,\underline{U}(s)$$

- **Rearrange**

$$s\,\underline{X}(s) - \underline{\underline{P}}\,\underline{X}(s) = \underline{x}(0^+) + \underline{\underline{B}}\,\underline{U}(s)$$

# State Transition Matrix

$$\underline{X}(s) = \left[s\underline{\underline{I}} - \underline{\underline{P}}\right]^{-1} \underline{x}(0^+) + \left[s\underline{\underline{I}} - \underline{\underline{P}}\right]^{-1} \underline{\underline{B}}\underline{U}(s)$$

- **Inverse Laplace transfer (the state transition equation)**

$$\underline{x}(t) = \underline{\underline{\Phi}}(t)\underline{x}(0^+) + \int_0^t \underline{\underline{\Phi}}(t-\tau)\underline{\underline{B}}\underline{u}(\tau)d\tau$$

- **The state transition matrix is defined as**

$$\underline{\underline{\Phi}}(t) = \mathscr{L}^{-1}\left\{\left[s\underline{\underline{I}} - \underline{\underline{P}}\right]^{-1}\right\}$$

# State Transition Matrix

- **Properties of state transition matrix**

$$\underline{\underline{\Phi}}(0) = \underline{\underline{I}}$$

$$\underline{\underline{\Phi}}(t_2 - t_0) = \underline{\underline{\Phi}}(t_2 - t_1)\underline{\underline{\Phi}}(t_1 - t_0)$$

$$\underline{\underline{\Phi}}(t + \tau) = \underline{\underline{\Phi}}(t)\underline{\underline{\Phi}}(\tau)$$

$$\underline{\underline{\Phi}}^{-1}(t) = \underline{\underline{\Phi}}(-t)$$

# State Transition Matrix

- **For more general initial time, recall**

$$\underline{x}(t) = \underline{\underline{\Phi}}(t)\underline{x}(0^+) + \int_0^t \underline{\underline{\Phi}}(t-\tau)\underline{\underline{B}}\underline{u}(\tau)d\tau$$

- **Rearrange and let** $t = t_0$

$$\underline{x}(t_0) = \underline{\underline{\Phi}}(t_0)\underline{x}(0^+) + \int_0^{t_0} \underline{\underline{\Phi}}(t_0-\tau)\underline{\underline{B}}\underline{u}(\tau)d\tau$$

$$\underline{\underline{\Phi}}^{-1}(t_0)\underline{x}(t_0) = \underline{x}(0^+) + \underline{\underline{\Phi}}^{-1}(t_0)\int_0^{t_0} \underline{\underline{\Phi}}(t_0-\tau)\underline{\underline{B}}\underline{u}(\tau)d\tau$$

$$\underline{x}(0^+) = \underline{\underline{\Phi}}^{-1}(t_0)\underline{x}(t_0) - \underline{\underline{\Phi}}^{-1}(t_0)\int_0^{t_0} \underline{\underline{\Phi}}(t_0-\tau)\underline{\underline{B}}\underline{u}(\tau)d\tau$$

**Pay attention to the order of the terms in matrix multiplication !**
**The—commutative law**

# State Transition Matrix

- **Substitute back to the state transition equation**

$$\underline{x}(t) = \underline{\underline{\Phi}}(t)\underline{\underline{\Phi}}(-t_0)\underline{x}(t_0) \\ -\underline{\underline{\Phi}}(t)\underline{\underline{\Phi}}(-t_0)\int_0^{t_0}\underline{\underline{\Phi}}(t_0-\tau)\underline{\underline{B}}\underline{u}(\tau)d\tau + \int_0^t \underline{\underline{\Phi}}(t-\tau)\underline{\underline{B}}\underline{u}(\tau)d\tau$$

- **Second term becomes**

$$-\underline{\underline{\Phi}}(t)\underline{\underline{\Phi}}(-t_0)\int_0^{t_0}\underline{\underline{\Phi}}(t_0-\tau)\underline{\underline{B}}\underline{u}(\tau)d\tau \\ = -\underline{\underline{\Phi}}(t-t_0)\int_0^{t_0}\underline{\underline{\Phi}}(t_0-\tau)\underline{\underline{B}}\underline{u}(\tau)d\tau \\ = \int_0^{t_0} -\underline{\underline{\Phi}}(t-t_0)\underline{\underline{\Phi}}(t_0-\tau)\underline{\underline{B}}\underline{u}(\tau)d\tau$$

# State Transition Matrix

- **and**

$$-\int_0^{t_0} \underline{\underline{\Phi}}(t-t_0)\underline{\underline{B}}\underline{u}(\tau)d\tau + \int_0^{t} \underline{\underline{\Phi}}(t-\tau)\underline{\underline{B}}\underline{u}(\tau)d\tau$$

$$= \int_{t_0}^{t} \underline{\underline{\Phi}}(t-\tau)\underline{\underline{B}}\underline{u}(\tau)d\tau$$

- **then**

$$\underline{x}(t) = \underline{\underline{\Phi}}(t-t_0)\underline{x}(t_0) + \int_{t_0}^{t} \underline{\underline{\Phi}}(t-\tau)\underline{\underline{B}}\underline{u}(\tau)d\tau$$

# State Transition Matrix

- **Example: an open loop system,**

$$P(s)=\frac{C(s)}{U(s)}=\frac{1}{s^2}$$

  - **Differential equation form is**

$$\ddot{c}(t)=u(t)$$

  - **Therefore, define the state variables**

$$x_1(t)=c(t) \qquad x_2(t)=\dot{c}(t)$$

# State Transition Matrix

**thus**

$$\begin{cases} \dot{x}_1(t) = x_2(t) = \dot{c}(t) \\ \dot{x}_2(t) = \ddot{c}(t) = u(t) \end{cases}$$

**in the phase-variable canonical form**

$$\underline{\dot{x}}(t) = \underline{\underline{P}}\,\underline{x}(t) + \underline{B}u(t)$$

$$\underline{\underline{P}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \underline{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \underline{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(x) \end{bmatrix} \quad \underline{\dot{x}}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(x) \end{bmatrix}$$

# State Transition Matrix

$$\left[s\underline{\underline{I}} - \underline{\underline{P}}\right] = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}$$

$$\left[s\underline{\underline{I}} - \underline{\underline{P}}\right]^{-1} = \frac{\operatorname{adj}\left[s\underline{\underline{I}} - \underline{\underline{P}}\right]}{\left|s\underline{\underline{I}} - \underline{\underline{P}}\right|} = \frac{\begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix}}{\begin{vmatrix} s & -1 \\ 0 & s \end{vmatrix}} = \frac{\begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix}}{s^2} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix}$$

– **The state transition matrix is,**

$$\underline{\underline{\Phi}}(t) = \mathscr{L}^{-1}\left\{\left[s\underline{\underline{I}} - \underline{\underline{P}}\right]^{-1}\right\} = \begin{bmatrix} U(t) & t \\ 0 & U(t) \end{bmatrix}$$

where $U(t)$ is the unit step function

# State Transition Matrix

– **Assume the initial conditions,**

$$\underline{x}(0^+) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

**and** $u(0) = 0$

$$\underline{x}(t) = \underline{\underline{\Phi}}(t)\underline{x}(0^+)$$

# State Transition Matrix

– **Therefore, (notice there is an error in the book)**

$$\underline{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} U(t) & t \\ 0 & U(t) \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

– **or**

$$x_1(t) = U(t) + 2t$$

$$x_2 = 2U(t)$$

# Total Solution of the State Equation

- **Example: a system describe by**

$$\ddot{c}(t) + 2\dot{c}(t) + c(t) = \dot{r}(t) + r(t)$$

- **Determine the output $c(t)$, given**

$$r(t) = \sin t$$

- **Initial conditions**

$$c(0) = 1 \qquad \dot{c}(0) = 0$$

# Total Solution of the State Equation

- **Determine the state transition matrix**

$$\underline{\underline{\Phi}}(t) = \mathscr{L}^{-1}\left\{\left[s\underline{\underline{I}} - \underline{\underline{P}}\right]^{-1}\right\}$$

- **Determine the output** $c(t)$

$$\underline{x}(t) = \underline{\underline{\Phi}}(t)\underline{x}(0^{+}) + \int_0^t \underline{\underline{\Phi}}(t-\tau)\underline{\underline{B}}\underline{u}(\tau)d\tau$$

$$\underline{c}(t) = \underline{\underline{L}}\underline{x}(t)$$

# Total Solution of the State Equation

- **Determine the state transition matrix**
  - **Define the state variable**

$$x_1(t) = c(t)$$

$$x_2(t) = \dot{c}(t)$$

**and we have**

$$u(t) = r(t)$$

$$\dot{u}(t) = \dot{r}(t)$$

# Total Solution of the State Equation

- First order differential equation representation of the system

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = 2x_2(t) - x_1(t) + u(t) + \dot{u}(t)$$

- The phase-variable canonical form is,

$$\dot{\underline{x}}(t) = \underline{\underline{P}}\,\underline{x}(t) + \underline{\underline{B}}\big(u(t) + \dot{u}(t)\big)$$

# Total Solution of the State Equation

**where**

$$\underline{\underline{P}} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \quad \underline{\underline{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \underline{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(x) \end{bmatrix} \quad \underline{\dot{x}}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(x) \end{bmatrix}$$

**and**

$$\left[s\underline{\underline{I}} - \underline{\underline{P}}\right] = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 1 & s+2 \end{bmatrix}$$

# Total Solution of the State Equation

– **Then**

$$\left[s\underline{\underline{I}}-\underline{\underline{P}}\right]^{-1}=\frac{\mathrm{adj}\left[s\underline{\underline{I}}-\underline{\underline{P}}\right]}{\left|s\underline{\underline{I}}-\underline{\underline{P}}\right|}=\frac{\begin{bmatrix} s+2 & 1 \\ -1 & s \end{bmatrix}}{\begin{vmatrix} s & -1 \\ 1 & s+2 \end{vmatrix}}=\frac{\begin{bmatrix} s+2 & 1 \\ -1 & s \end{bmatrix}}{(s+1)^2}=\begin{bmatrix} \frac{s+2}{(s+1)^2} & \frac{1}{(s+1)^2} \\ -\frac{1}{(s+1)^2} & \frac{s}{(s+1)^2} \end{bmatrix}$$

– **Therefore, the state transition matrix is**

$$\underline{\underline{\Phi}}(t)=\mathscr{L}^{-1}\left\{\left[s\underline{\underline{I}}-\underline{\underline{P}}\right]^{-1}\right\}=\begin{bmatrix} (1+t)e^{-t} & te^{-t} \\ -te^{-t} & (1-t) \end{bmatrix}$$

# Total Solution of the State Equation

- **Determine the output** $c(t)$

$$\underline{x}(t) = \underline{\underline{\Phi}}(t)\underline{x}(0^+) + \int_0^t \underline{\underline{\Phi}}(t-\tau)\underline{\underline{B}}\underline{u}(\tau)d\tau$$

**Substitute $\underline{\boldsymbol{x}}(\boldsymbol{t})$ into**

$$\underline{c}(t) = \underline{\underline{L}}\underline{x}(t)$$

**result in**

$$\underline{c}(t) = \underline{\underline{L}}\underline{\underline{\Phi}}(t)\underline{x}(0^+) + \int_0^t \underline{\underline{L}}\underline{\underline{\Phi}}(t-\tau)\underline{\underline{B}}\underline{u}(\tau)d\tau$$

# Total Solution of the State Equation

**from**

$$x_1(t) = c(t)$$

$$x_2(t) = \dot{c}(t)$$

**and given**

$$c(0) = 1 \qquad \dot{c}(0) = 0$$

**we have**

$$\underline{\underline{L}} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\underline{x}(0^+) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} c(0) \\ \dot{c}(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

# Total Solution of the State Equation

**at the same time, given**

$$u(t) = r(t) = \sin t$$

**Therefore**

$$\dot{u}(\tau) + u(\tau) = \dot{r}(\tau) + r(\tau) = \sin\tau + \cos\tau$$

**and**

$$\underline{\underline{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

# Total Solution of the State Equation

**Substitute all of them into** $c(t)$ **we have**,

$$\underline{c}(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} (1+t)e^{-t} & e^{-t} \\ e^{-t} & (1-t)e^{-t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$+ \int_0^t \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} (1+t-\tau)e^{-(t-\tau)} & e^{-(t-\tau)} \\ e^{-(t-\tau)} & (1-t+\tau)e^{-(t-\tau)} \end{bmatrix}$$

$$\times \begin{bmatrix} 0 \\ 1 \end{bmatrix} (\sin\tau + \cos\tau)\, d\tau$$

# Total Solution of the State Equation

## On simplifying

$$c(t) = e^{-t}(t+1) + \int_0^t \left[(t-\tau)e^{-(t-\tau)}\right](\sin\tau + \cos\tau)\,d\tau$$

$$= \frac{3}{2}e^{-t} + te^{-t} + \frac{1}{2}\sin t - \frac{1}{2}\cos t$$

## check the initial conditions

$$c(0) = \frac{3}{2} + 0 + 0 - \frac{1}{2}\times 1 = 1$$

$$\dot{c}(0) = -\frac{3}{2} + 0 + 1 + \frac{1}{2}\times 1 + 0 = 0$$

# Homework V-1

**A–9–7.** Obtain the response $y(t)$ of the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} -1 & -0.5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} u, \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where $u(t)$ is the unit-step input occurring at $t = 0$, or

$$u(t) = 1(t)$$

# Homework V-2

**2.75.** Substances $x_1(t)$ and $x_2(t)$ are involved in the reaction of a chemical process. The state equations representing this reaction are as follows:

$$\dot{x}_1(t) = -4x_1(t) + 2x_2(t),$$
$$\dot{x}_2(t) = 2x_1(t) - x_2(t).$$

**(a)** Determine the state transition matrix of this chemical process.

**(b)** Determine the response of this system when:

$$x_1(0) = 200{,}000 \text{ units},$$
$$x_2(0) = 10{,}000 \text{ units}.$$

**(c)** At what value of time will the amount of substances $x_1(t)$ and $x_2(t)$ be equal?