Math Review

- **Quadratic equation**

$$\alpha s^2 + \beta s + 1 = 0$$

The roots are:

$$s = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha}}{2\alpha}$$

Discriminant:

$$\beta^2 - 4\alpha$$

Math Review

- **Complex number**

$$s = \sigma + j\omega$$

- **Complex function**

$$F(s) = \text{Real}\, F(s) + j\ \text{Imaginary}\, F(s)$$

- **Most complex functions in linear control systems are single-valued functions of s**

Math Review

- **Singularities, Poles and Zeros**

$$F(s) = \frac{100(s+1)(s+8)^2}{s(s+4)(s+10)(s+20)^2}$$

  – **Poles (denominator):**

  s = 0, -4, -10 and *s* = -20 (second order)

  – **Zeros (numerator):**

  s = -1 and s = -8 (second order)

Review of Fourier Series and Fourier Transform

- **Classical trigonometric form of the Fourier series:**

$$f(t) = \frac{A_0}{2} + \sum_{k=1}^{k=\infty} A_k \cos k\omega t + \sum_{k=1}^{k=\infty} B_k \sin k\omega t$$

$$A_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos k\omega t \, dt \; , \; k = 1, 2, 3, \ldots,$$

$$B_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin k\omega t \, dt \; , \; k = 1, 2, 3, \ldots.$$

Complex Form of the Fourier Series

- **Using substitutions:**

$$\sin k\omega t = \frac{1}{2j}(e^{jk\omega t} - e^{-k\omega t})$$

$$\cos k\omega t = \frac{1}{2j}(e^{jk\omega t} + e^{-k\omega t})$$

then,

$$f(t) = \frac{A_0}{2} + \frac{1}{2}\sum_{k=1}^{\infty}(A_k - jB_k)e^{jk\omega t} + \frac{1}{2}\sum_{k=1}^{\infty}(A_k + jB_k)e^{jk\omega t}$$

Complex Form of the Fourier Series

$$f(t) = \frac{A_0}{2} + \sum_{k=1}^{\infty} \frac{e^{jk\omega t}}{T} \int_{-T/2}^{T/2} f(t) e^{-jk\omega t} dt$$

$$+ \sum_{k=1}^{\infty} \frac{e^{-jk\omega t}}{T} \int_{-T/2}^{T/2} f(t) e^{jk\omega t} dt$$

therefore,

$$f(t) = \sum_{k=-\infty}^{\infty} \frac{e^{jk\omega t}}{T} \int_{-T/2}^{T/2} f(t) e^{-jk\omega t} dt$$

Complex Form of the Fourier Series

- **In the more traditional form,**

$$f(t) = \frac{1}{T} \sum_{k=-\infty}^{k=\infty} C_k e^{jk\omega t}$$

where

$$C_k = \int_{-T/2}^{T/2} f(t) e^{-jk\omega t} dt$$

Fourier Integral/Transform

- **For functions that are not periodic, the Fourier series cannot be applied. Therefore we have to introduce Fourier integral by assuming they are periodic with a period of infinity.**

Let $$T = 2\pi / \Delta k\omega$$

then $$f(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{k=\infty} C_k e^{jk\omega t} \Delta k\omega$$

Fourier Integral/Transform

$$\lim_{\substack{T\to\infty \\ \Delta k\omega\to 0}} f(t) = \lim_{\substack{T\to\infty \\ \Delta k\omega\to 0}} \frac{1}{2\pi} \sum_{k=-\infty}^{k=\infty} C_k e^{jk\omega t} \Delta k\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} C_k e^{j\omega t} dt$$

where $$C_k = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$= F(\omega)$$

This is the definition of the famous Fourier Transform !

Fourier Integral/Transform

- **But there is a serious limitation to the transformation, the integral must be bounded**

$$\int_{-\infty}^{\infty} \left| f(t) \right| dt < \infty$$

- **For unit step, ramp and parabolic time functions the transformation do not exist, however they are exactly what we need!**

Laplace Transform

- **In order to force functions into absolute convergence, we have to introduce a damping factor into the equation**

$$F(\sigma,\omega)=\int_0^{\infty} f(t)e^{-\sigma t}e^{-j\omega t}dt$$

and let $s=\sigma+j\omega$

We have $$F(s)=\int_0^{\infty} f(t)e^{-st}dt$$

This is the definition of the famous Laplace Transform !

Laplace Transform

- **Traditionally we use the following form,**

$$\mathcal{L}[f(t)] = F(s) = \int_0^{\infty} f(t)e^{-st}dt$$

- **Useful Laplace Transform**
  - **Step function**

$$\mathcal{L}[a] = \int_0^{\infty} ae^{-st}dt = -\frac{a}{s} \cdot e^{-st}\Big|_0^{\infty} = \frac{a}{s}$$

  - **Unit step** $\mathcal{L}[1] = \frac{1}{s}$

Useful Laplace Transform

– **Unit ramp**

$$\mathcal{L}[t] = \int_0^\infty te^{-st}dt$$

- **Integrating by parts,**

$$\int u\,dv = uv - \int v\,du$$

- **Let** $u = t$ **and** $dv = e^{-st}dt$

$$\int_0^\infty te^{-st}dt = \left(t\frac{e^{-st}}{-s} \right)\Bigg|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}dt = \frac{1}{s^2}$$

Useful Laplace Transform

– **Exponential Decay**

$$\mathcal{L}\left[e^{-\alpha t}\right] = \int_0^\infty e^{-\alpha t} e^{-st} dt$$

$$= \left(-\frac{1}{s+\alpha} e^{-(s+\alpha)t} \right)\Bigg|_0^\infty$$

$$= \frac{1}{s+\alpha}$$

Properties of the Laplace Transform

– **Addition and Subtraction**

$$\mathscr{L}\left[f_1(t) \pm f_2(t)\right] = F_1(s) \pm F_2(s)$$

– **Multiplication by a Constant**

$$\mathscr{L}\left[k \cdot f(t)\right] = k \cdot F(s)$$

Properties of the Laplace Transform

– Direct Transform of Derivatives

$$\mathscr{L}\left[\frac{df}{dt}\right] = \int_0^\infty \frac{df}{dt} e^{-st} dt = sF(s) - f(0^+)$$

$$\mathscr{L}\left[\frac{d^2 f}{dt^2}\right] = \mathscr{L}\left[\mathscr{L}\left[\frac{df}{dt}\right]\right] = s^2 F(s) - sf(0^+) - \dot{f}(0^+)$$

$$\mathscr{L}\left\{\frac{d^n f}{dt^n}\right\} = s^n F(s) - s^{n-1} f(0^+) - s^{n-2} \dot{f}(0^+) - \cdots - f^{(n-1)}(0^+)$$

Properties of the Laplace Transform

– Direct Transform of Integrals

$$\mathscr{L}\left[\int f(\tau)d\tau\right] = \int_0^{\infty}\left(\int f(\tau)d\tau\right)e^{-st}dt$$

$$= \int_0^{\infty}\frac{1}{s}f(t)e^{-st}dt + \left(\int f(\tau)d\tau\right)\left[-\frac{1}{s}e^{-st}\right]\Bigg|_{t=0}^{\infty}$$

$$= \frac{F(s)}{s} + \frac{1}{s}\left[\int f(\tau)d\tau\right]_{t=0^+}$$

Properties of the Laplace Transform

– **Do the following as homework**

$$\mathscr{L}\left[\int\int f(t)dt^2\right] = \frac{F(s)}{s^2} + \frac{1}{s^2}\left[\int f(t)dt\right]_{t=0^+} + \frac{1}{s}\left[\int\int f(t)dt^2\right]_{t=0^+}$$

$$\mathscr{L}\left[\int\int\cdots\int f(t)dt^n\right] = \frac{F(s)}{s^n} + \frac{1}{s^n}\left[\int f(t)dt\right]_{t=0^+} + \frac{1}{s^{n-1}}\left[\int\int f(t)dt^2\right]_{t=0^+}$$

$$+\cdots+\frac{1}{s}\left[\int\int\cdots\int f(t)dt^n\right]_{t=0^+}$$

Properties of the Laplace Transform

– **Time-Shifting Theorem**

$$\mathscr{L}\left[f(t-\tau_d)U(t-\tau_d)\right]=e^{-st}F(s) \quad \mathrm{t} \geq \tau_d$$

– **Frequency-Shifting Theorem**

$$\mathscr{L}\left[e^{-\alpha t} f(t)\right]=\int_0^\infty e^{-\alpha t} f(t)e^{-st}dt = F(s+\alpha)$$

– **Initial-Value Theorem**

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

– **Final-Value Theorem**

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Inverse Laplace Transform

- **Definition of inverse Laplace Transform**

$$\mathscr{L}^{-1}F(s) = f(t)$$

- **Inversion by partial fraction expansion**
  - **Example I**

$$F(s) = \frac{As+B}{(s+C)(s+D)}$$

$$= \frac{B-AC}{D-C}\frac{1}{s+C} + \frac{B-AD}{C-D}\frac{1}{s+D}$$

$$f(t) = \frac{B-AC}{D-C}e^{-Ct} + \frac{B-AD}{C-D}e^{-Dt}$$

Inverse Laplace Transform

– **Example II**

$$F(s)=\frac{As+B}{(s+C)^2(s+D)}$$

$$=\frac{B-AC}{D-C}\frac{1}{(s+C)^2}+\frac{AD-B}{(D-C)^2}\frac{1}{s+C}+\frac{B-AD}{(C-D)^2}\frac{1}{s+D}$$

$$f(t)=\frac{B-AC}{D-C}te^{-Ct}+\frac{AD-B}{(D-C)^2}e^{-Ct}+\frac{B-AD}{(C-D)^2}e^{-Dt}$$

Laplace-Transform Solution of Differential Equations

- **Linear ordinary differential equation (ODE)**

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = 6$$

$$\dot{y}(0^+) = 2, \qquad y(0^+) = 2$$

  - **Laplace transform**

$$s^2Y(s) - sy(0^+) - \dot{y}(0^+) + 5sY(s) - 5y(0^+) + 6Y(s) = \frac{6}{s}$$

Laplace-Transform Solution of Differential Equations

– **Rearrange**

$$Y(s) = \frac{2s^2 + 12s + 6}{s(s^2 + 5s + 6)} = \frac{2s^2 + 12s + 6}{s(s+2)(s+3)}$$

– **Partial fraction expansion**

$$Y(s) = \frac{1}{s} + \frac{5}{s+2} + \frac{4}{s+3}$$

Laplace-Transform Solution of Differential Equations

– **Inverse Laplace transform**

$$y(t) = 1 + 5e^{-2t} - 4e^{-3t}$$

– **Final-value theorem check**

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

$$\lim_{s \to 0} s \frac{2s^2 + 12s + 6}{s(s+2)(s+3)} = 1$$

Laplace-Transform Solution of Differential Equations

– **Non-analytic ODE**

$$\frac{d^2y(t)}{dt^2}+\frac{dy(t)}{dt}=e^{4t}$$

$$y(0^+)=2, \quad \dot{y}(0^+)=0$$

– **Laplace transform**

$$s^2Y(s)-sy(0^+)-\dot{y}(0^+)+sY(s)-y(0^+)=\frac{1}{s-4}$$

Ordinary Differential Equation

– **Rearrange**

$$Y(s) = \frac{2s^2 - 6s - 7}{s(s+1)(s-4)}$$

– **Partial fraction expansion**

$$Y(s) = \frac{\frac{7}{4}}{s} + \frac{\frac{1}{5}}{s+1} + \frac{\frac{1}{20}}{s-4}$$

Ordinary Differential Equation

– **Inverse Laplace transform**

$$y(t) = 1.75 + \frac{1}{5}e^{-t} + \frac{1}{20}e^{4t}$$

– **Final-value theorem check**

$$\lim_{s\to 0} sY(s) = s\,\frac{2s^2 - 6s - 7}{s(s+1)(s-4)} = \frac{7}{4}$$

$$\lim_{t\to\infty} y(t) = 1.75 + \frac{1}{5}e^{-t} + \frac{1}{20}e^{4t} = \infty$$

$$\lim_{t\to\infty} f(t) \neq \lim_{s\to 0} sF(s)$$