Lead-Lag Compensation

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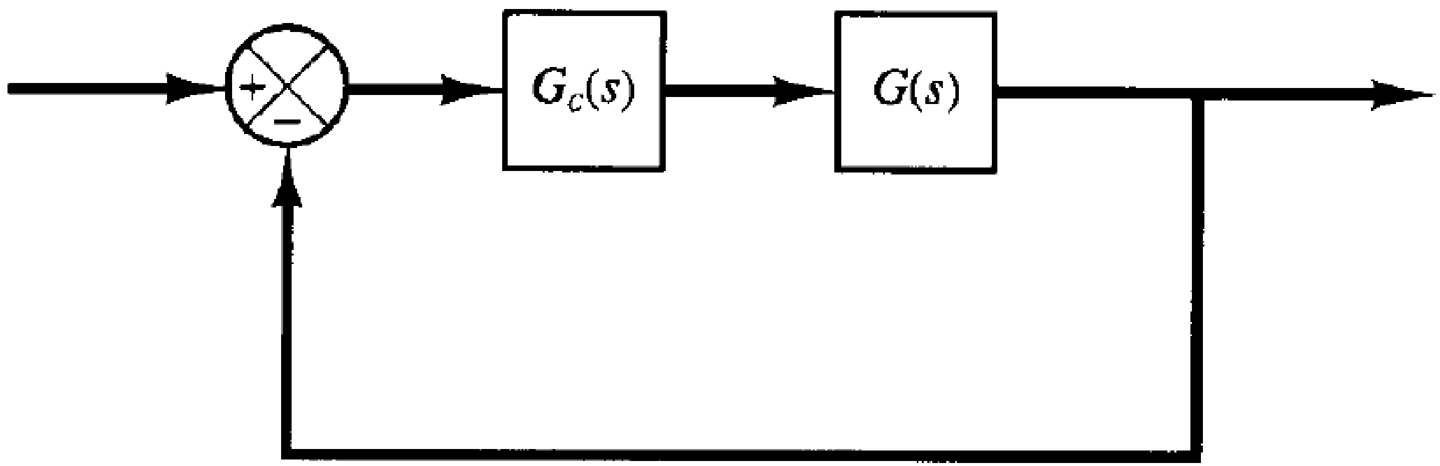
“lead” and “lag”

- **Adding a pole to the system changes the phase by -90 deg and adding a zero changes the phase by +90 deg.**
- **Generally the purpose of the Lead-Lag compensator is to create a controller which has an overall magnitude of approximately 1.**
- **It is used for phase compensation rather than magnitude.**

Lead Compensator

- **Assume the following lead compensator is :**

$$G_c(s) = K_c\alpha \frac{Ts+1}{\alpha Ts+1} = K_c \frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}} \qquad 0 < \alpha < 1$$



- **where** $G(s) = \dfrac{4}{s(s+2)}$
- **It is desired to design a compensator for the system so that the static velocity error constant K_v is 20 sec $^{-1}$, the phase margin is at least 50^o, and the gain margin is at least 10 dB.**
- **Define** $K_c \alpha = K$
- **Then** $G_c(s) = K \dfrac{Ts+1}{\alpha Ts+1}$

- **The open-loop transfer function of the system is**

$$G_c(s)G(s) = K\frac{Ts+1}{\alpha Ts+1}G(s) = \frac{Ts+1}{\alpha Ts+1}G_1(s)$$

- **and** $G_1(s) = KG(s) = \dfrac{4K}{s(s+2)}$

- **therefore,** $K_v = \lim_{s\to 0} sG_c(s)G(s) = \lim_{s\to 0} s\dfrac{Ts+1}{\alpha Ts+1}G_1(s)$

$$= \lim_{s\to 0}\frac{s4K}{s(s+2)} = 2K = 20$$

- **Next plot the bode diagram of**

$$G_1(j\omega) = KG(j\omega) = \frac{40}{j\omega(j\omega+2)} = \frac{20}{j\omega(0.5j\omega+1)}$$

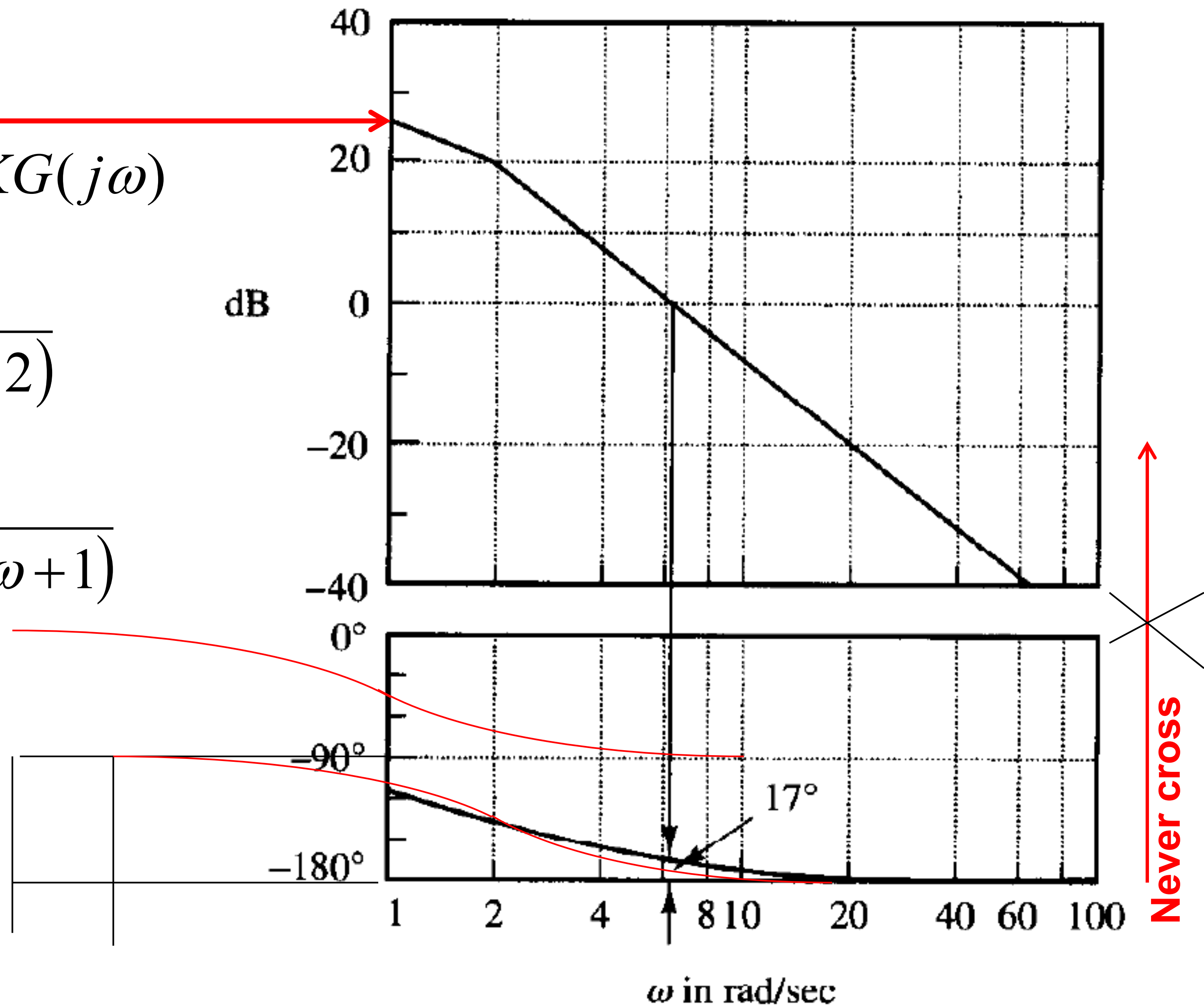
- **The transfer function has 3 components:**
  - **A constant of 20—which is equal to 26.0 dB. The phase is constant at 0 degrees.**
  - **A pole at s=0—break at 1rad/sec, a pure integral with slope -20dB/decade**
  - **The phase is 0 degrees up to 1/10 the break frequency (0.1 rad/sec) then drops linearly down to -90 degrees at 10 times the break frequency (10 rad/sec).**

- **A pole at s=-2—break at 2rad/sec, with slope of (-20)+(-20)dB/decade**
- **The phase is 0 degrees up to 1/10 the break frequency (0.2 rad/sec) then drops linearly down to -180 degrees at 10 times the break frequency (20 rad/sec).**

$$G_1(j\omega) = KG(j\omega)$$

$$= \frac{40}{j\omega(j\omega+2)}$$

$$= \frac{20}{j\omega(0.5j\omega+1)}$$



- **The phase margin of the system is 17^o and the gain margin is $+\infty$dB**
- **Phase margin of 17^o implies that the system is quite oscillatory.**
- **The specification calls for a phase margin of at least 50^o.**
- **A lead compensator is needed for this purpose.**

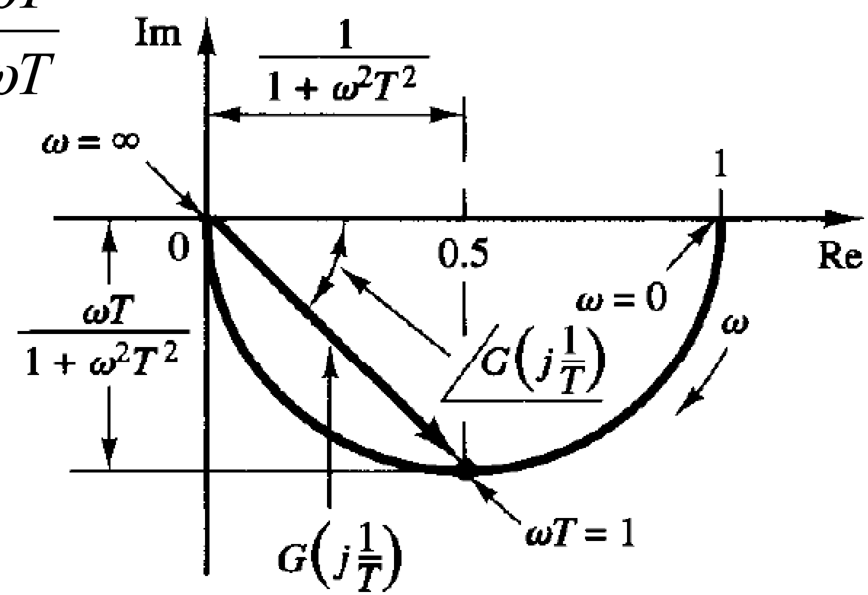
Polar Plot

- **Consider the Nyquist plots of the First order factor** $(1+s)^{-1} \rightarrow (1+j\omega)^{-1}$**.**
- **The sinusoidal transfer function**

$$G(j\omega)=\frac{1}{1+j\omega T}=\frac{1}{1+j\omega T}\frac{1-j\omega T}{1-j\omega T}$$

$$=\frac{1}{1+\omega^2T^2}+j\frac{-\omega T}{1+\omega^2T^2}$$

$$=\frac{1}{\sqrt{1+\omega^2T^2}}\left(-\arctan(\omega T)\right)$$

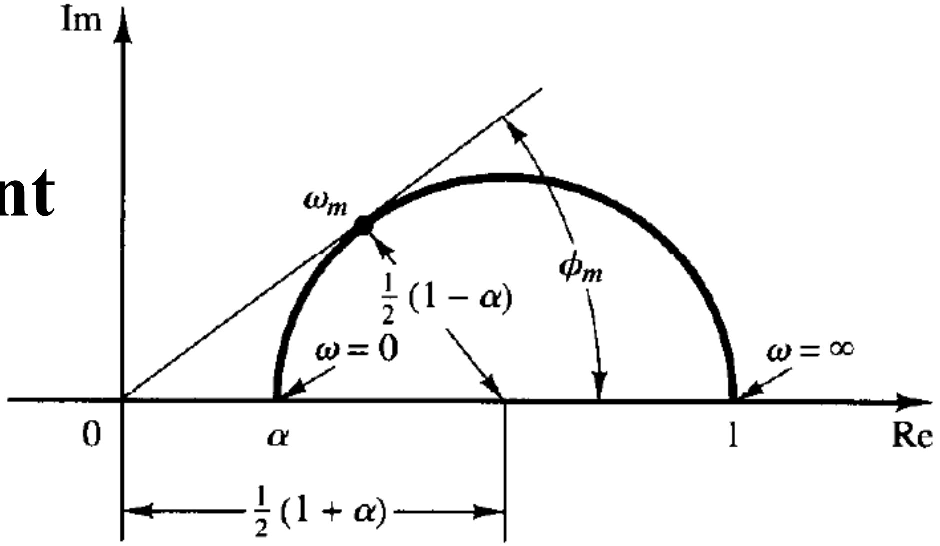


similarly

- **Polar plot of**

$$G_c(s) = K_c \alpha \frac{Ts+1}{\alpha Ts+1} = K_c \frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}} \qquad 0 < \alpha < 1$$

- **With $K_c = 1$, for a given value α, the angle between the positive real axis and the tangent line drawn from the origin to the semicircle gives the maximum phase lead angle.**



- **Phase angle at $\omega = \omega_m$ is φ_m, where**

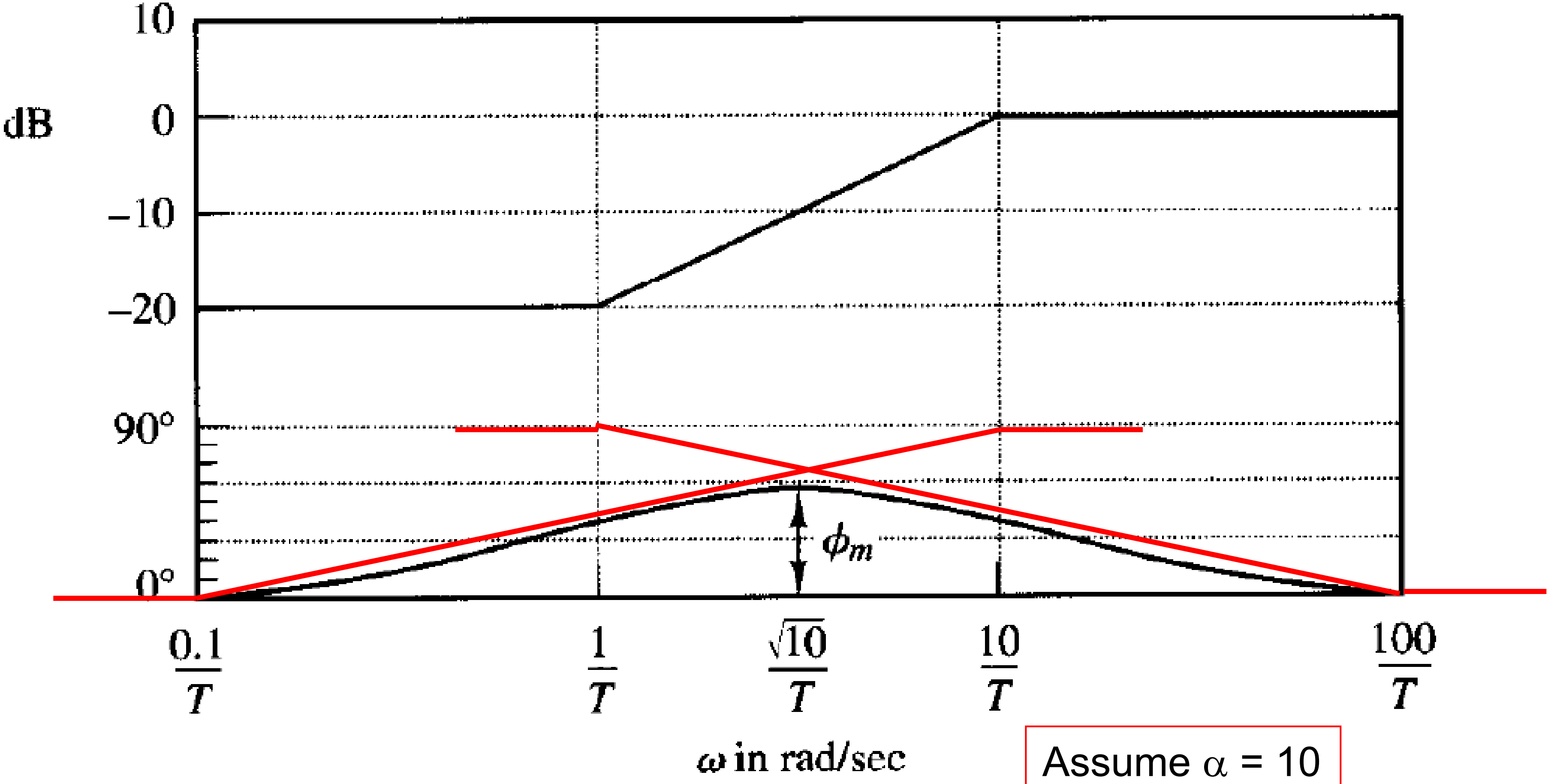
$$\sin\varphi_m = \frac{\frac{1-\alpha}{2}}{\frac{1+\alpha}{2}} = \frac{1-\alpha}{1+\alpha}$$

- **and ω_m is the geometric mean of two corner frequencies,**

$$\log\omega_m = \frac{1}{2}\left(\log\frac{1}{T} + \log\frac{1}{\alpha T}\right)$$

$$\omega_m = \frac{1}{\sqrt{\alpha}T}$$

Bode Diagram of Lead Compensator



Assume α = 10

- **To have a phase margin of at least 50°, phase lead of 33° is needed**
- **Notice that the addition of a lead compensator, will cause the gain crossover frequency shift to the right.**
- **We must offset the increased phase lag of $G_1(j\omega)$ due to this increase.**
- **Assume the $\varphi_m = 38^o$, therefore 5^o is used for the offset**

- **Since** $\sin\varphi_m = \dfrac{1-\alpha}{1+\alpha} = \sin 38^o$

- **Therefore, α = 0.24.**
- **Next**

$$\left|G_c(j\omega)\right| = \left|\frac{1+j\omega T}{1+j\omega\alpha T}\right|_{\omega=\frac{1}{\sqrt{\alpha}T}} = \left|\frac{1+j\frac{1}{\sqrt{\alpha}}}{1+j\alpha\frac{1}{\sqrt{\alpha}}}\right| = \frac{1}{\sqrt{\alpha}} = 6.2dB$$

- **Let** $\left|G_1(j\omega)\right| = -6.2dB$ **, therefore ω_c = 9 rad/sec.**

crossover

- **Therefore,** $$\frac{1}{T} = \sqrt{\alpha}\,\omega_c = 4.41$$

$$\frac{1}{\alpha T} = \frac{\omega_c}{\sqrt{\alpha}} = 18.4$$

- **The lead compensator thus determined is**

$$G_c = K_c \frac{s + 4.41}{s + 18.4} = K_c \alpha \frac{0.227s + 1}{0.054s + 1}$$

- **where** $$K_c = \frac{K}{\alpha} = \frac{10}{0.24} = 41.7$$

- **Thus, the transfer function of the compensator is**

$$G_c = 41.7\frac{s+4.41}{s+18.4} = 10\frac{0.227s+1}{0.054s+1}$$

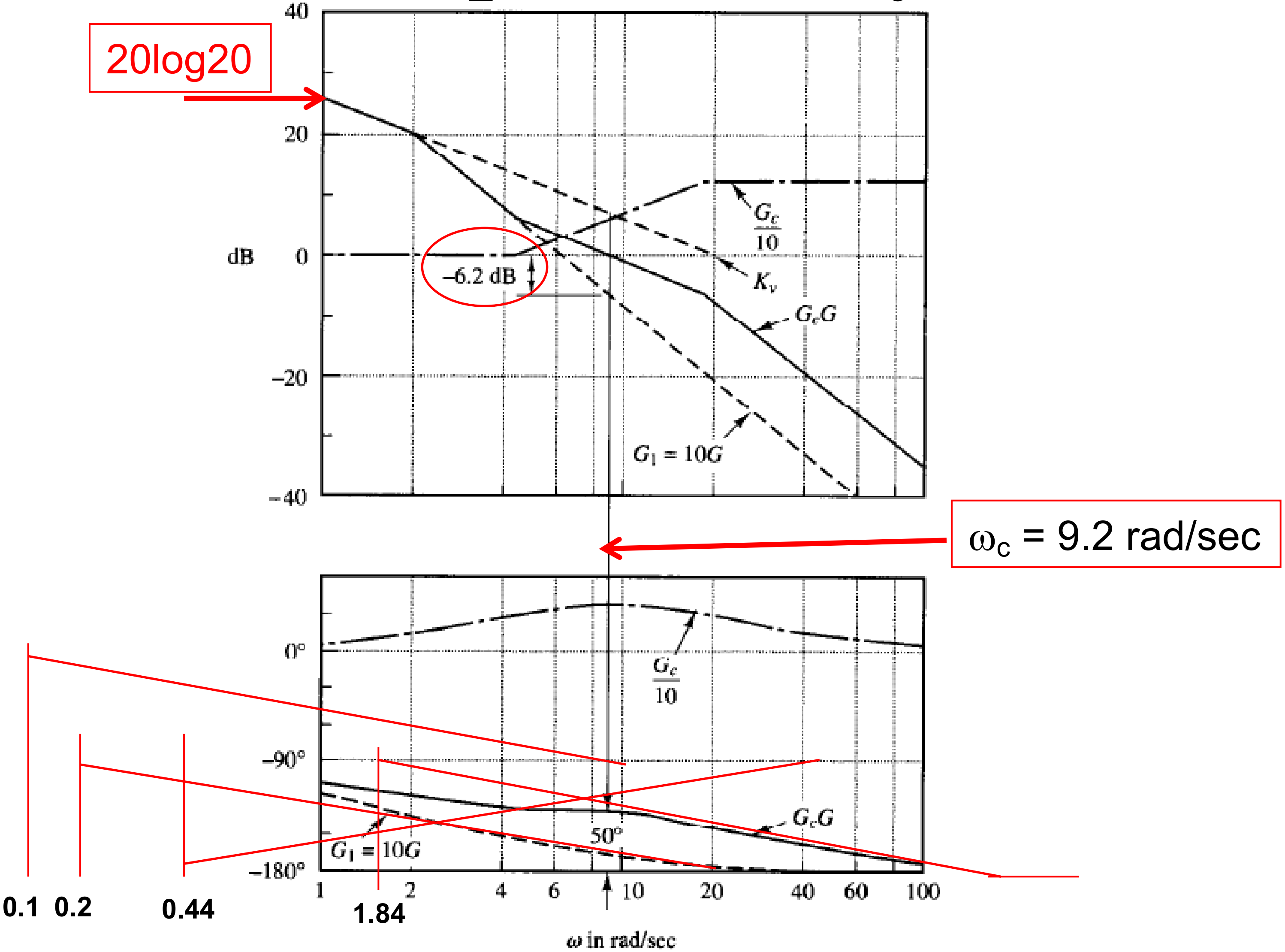
- **and**

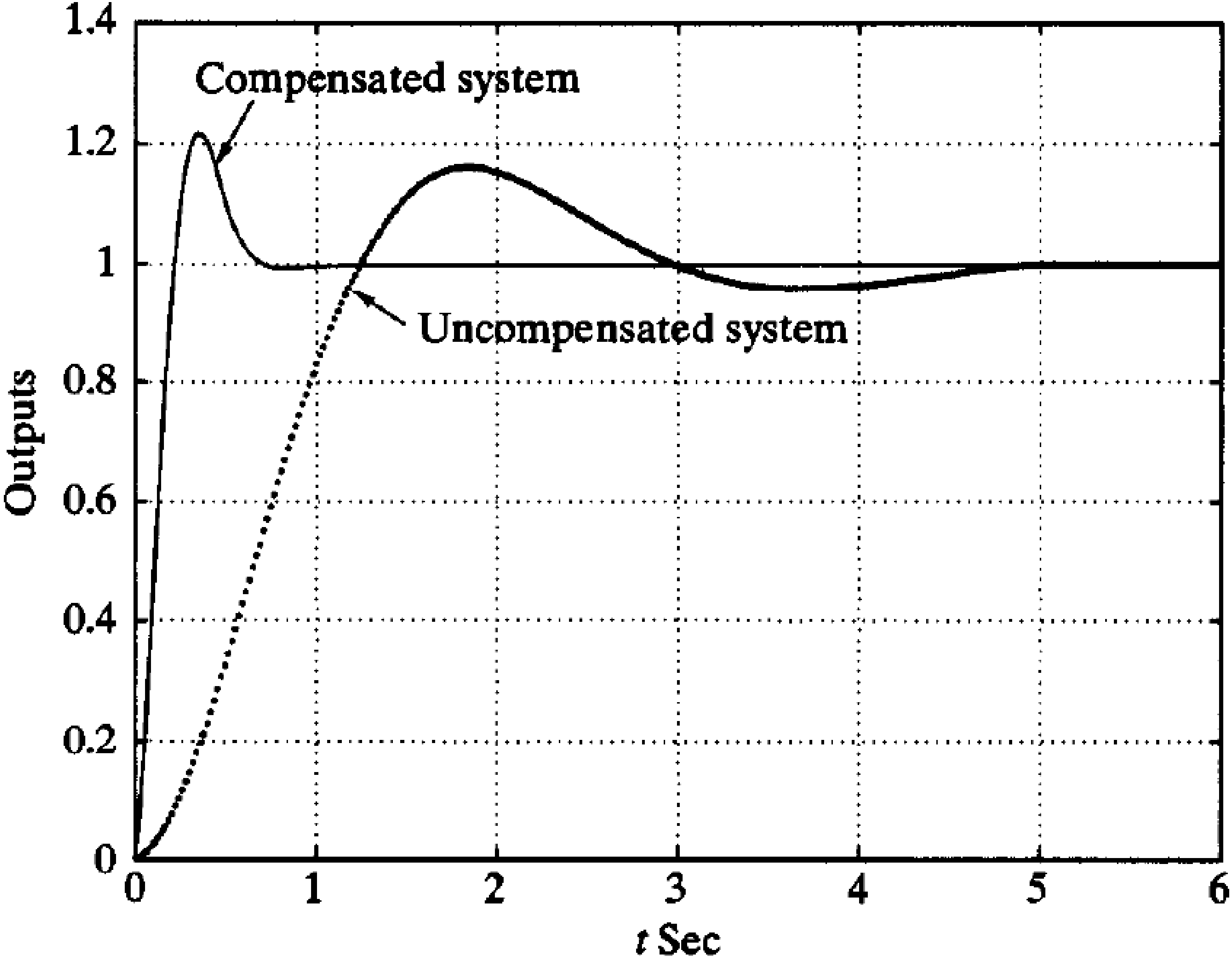
$$\frac{G_c(s)}{K}G_1(s) = \frac{G_c(s)}{10}10G(s) = G_c(s)G(s)$$

$$G_c(s)G(s) = 41.7\frac{(s+4.41)}{(s+18.4)}\frac{4}{s(s+2)}$$

$$G_c(s)G(s) = 10\frac{(.227s+1)}{(0.054s+1)}\frac{2}{s(0.5s+1)}$$

Bode Diagram of the Compensated System




Compensated system
Uncompensated system
Outputs
t Sec
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
1
2
3
4
5
6

Steady state error of cascade compensator

$$G_c(s) = \frac{1+\alpha Ts}{1+Ts} \approx 1+\alpha Ts$$

$$G(s) = \frac{\omega_n^2}{s(s+2\varsigma\omega_n)}$$

$$\frac{C(s)}{R(s))} = \frac{G_C G}{1+G_C G}$$

$$= \frac{\omega_n^2 + \alpha T\omega_n^2 s}{s^2 + \left(2\xi\omega_n + \alpha T\omega_n^2 Ts\right) + \omega_n^2}$$

$$2\zeta\omega_n + \alpha T\omega_n^2 = 2\zeta_{eq}\omega_n$$

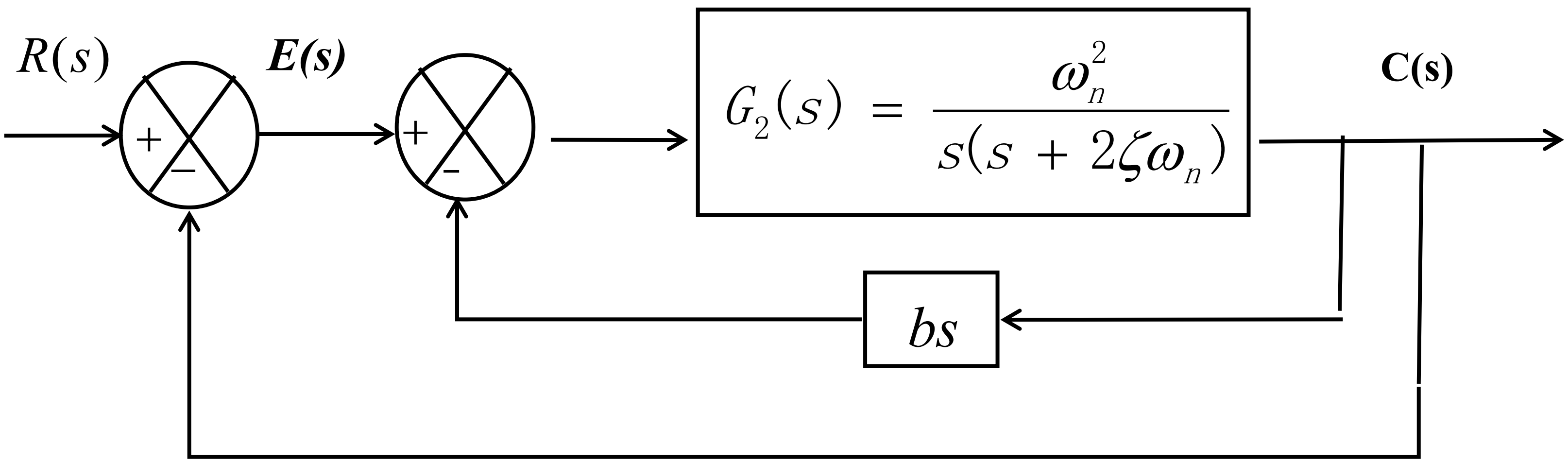
$$\zeta_{eq} = \zeta + \frac{\alpha T\omega_n}{2}$$

$R(s) = 1/s^2$, **unit ramp input**

$$E(s) = \frac{1}{s}\frac{(s + 2\zeta\omega_n)(1 + Ts)}{s(s + 2\zeta\omega_n)(1 + Ts) + \omega_n^2(1 + \alpha Ts)}$$

$$e_{ss(ramp\ input)} = \lim_{s\to 0} sE(s) = \frac{2\zeta}{\omega_n}$$

Minor- loop Feedback Compensation



$$G(s) = \frac{\omega_n^2}{s\left(s + 2\varsigma\omega_n\right)}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + \left(2\xi\omega_n + \omega_n^2 bs\right) + \omega_n^2}$$

$$2\zeta\omega_n + \omega_n^2 b = 2\zeta_{eq}\omega_n$$

$$\zeta_{eq} = \zeta + \frac{\omega_n b}{2}$$

$R(s) = 1 / s^2$, **unit ramp input**

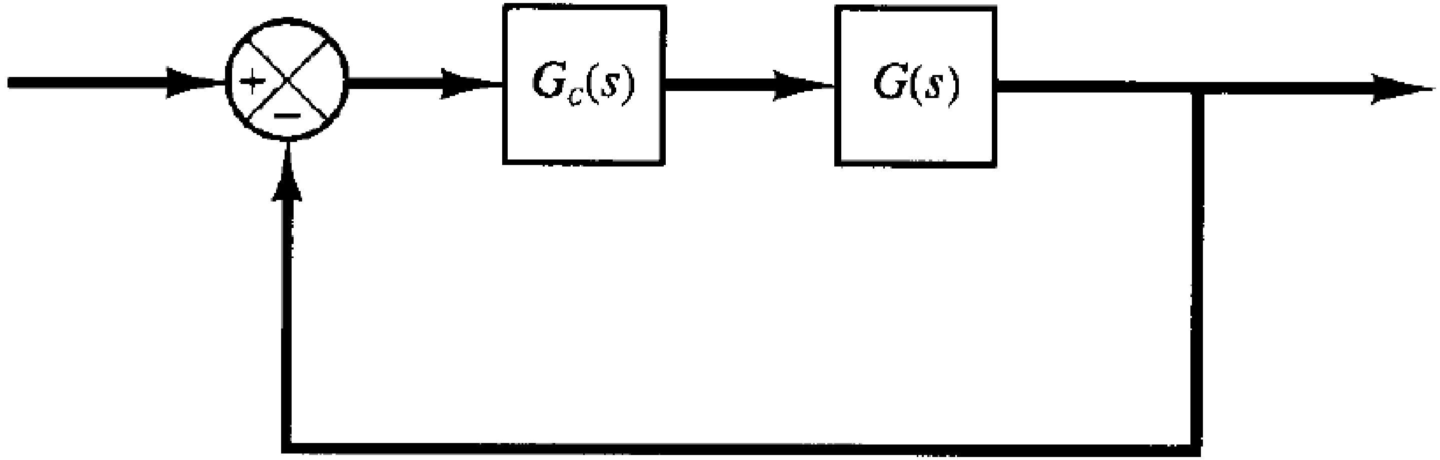
$$E(s) = \frac{1}{s}\left(\frac{s + 2\zeta\omega_n + b\omega_n^2}{s\left(s + 2\zeta\omega_n + b\omega_n^2\right) + \omega_n^2}\right)$$

$$e_{ss(ramp\ input)} = \lim_{s \to 0} sE(s) = \frac{2\zeta\omega_n + b\omega_n^2}{\omega_n^2} = \frac{2\zeta}{\omega_n} + b$$

Lag Compensator

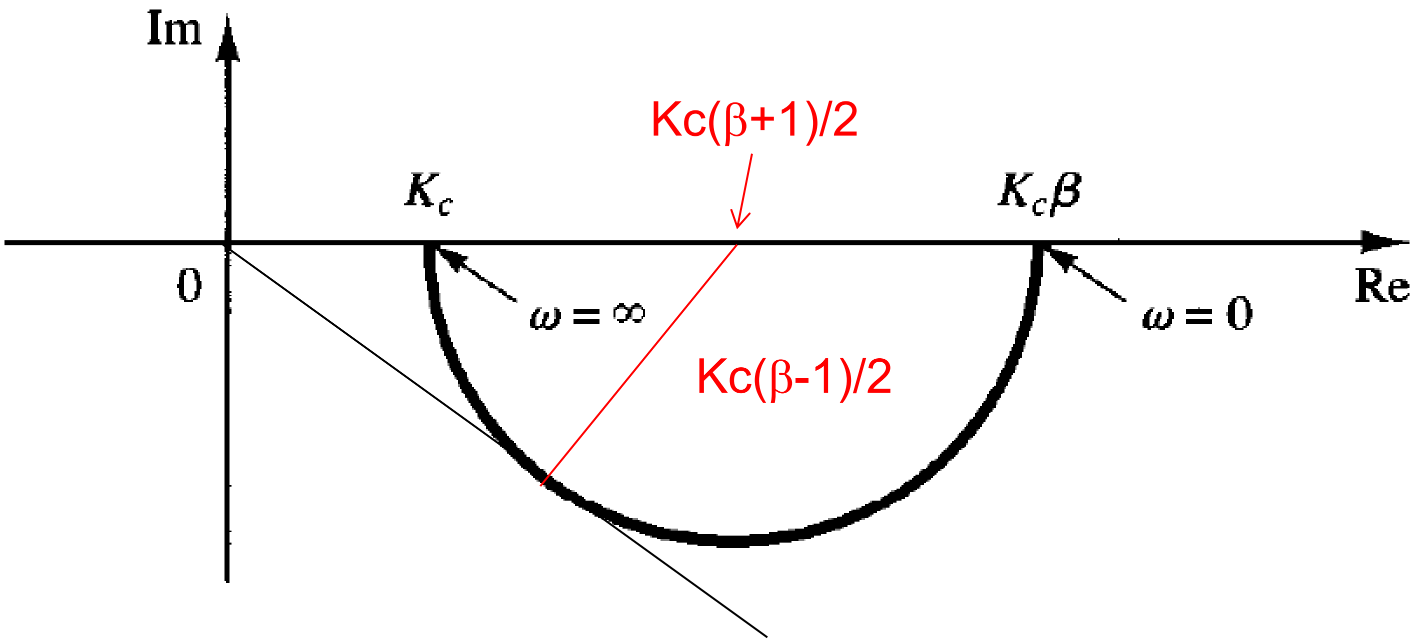
- **Assume the following lag compensator is :**

$$G_c(s) = K_c\beta\frac{Ts+1}{\beta Ts+1} = K_c\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}} \qquad \beta > 1$$

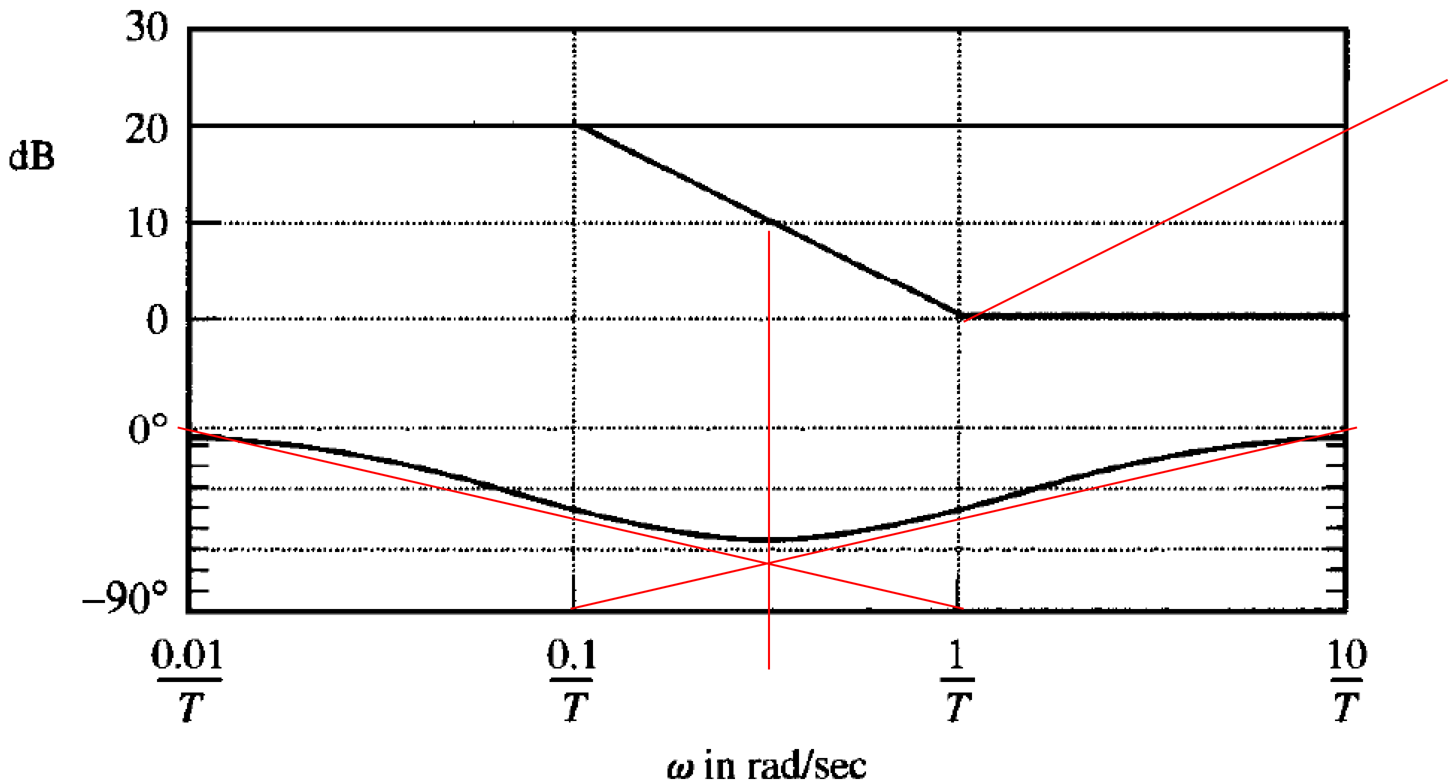


Lag Compensator

- the polar plot is given as:



$$G_c(s) = K_c \beta \frac{Ts+1}{\beta Ts+1}, \qquad assume \quad K_c = 1, \beta = 10$$



- **The open-loop transfer function of the system is**

$$G_c(s)G(s) = \frac{Ts+1}{\beta Ts+1}KG(s) = \frac{Ts+1}{\beta Ts+1}G_1(s)$$

- **Define** $K_c\beta = K$, $G(s) = \dfrac{1}{s(s+1)(0.5s+1)}$

- **It is desired to compensate the system so that the static velocity error constant K_v is 5 sec^{-1}, the phase margin is at least 40^o, gain margin is at least 10 dB.**

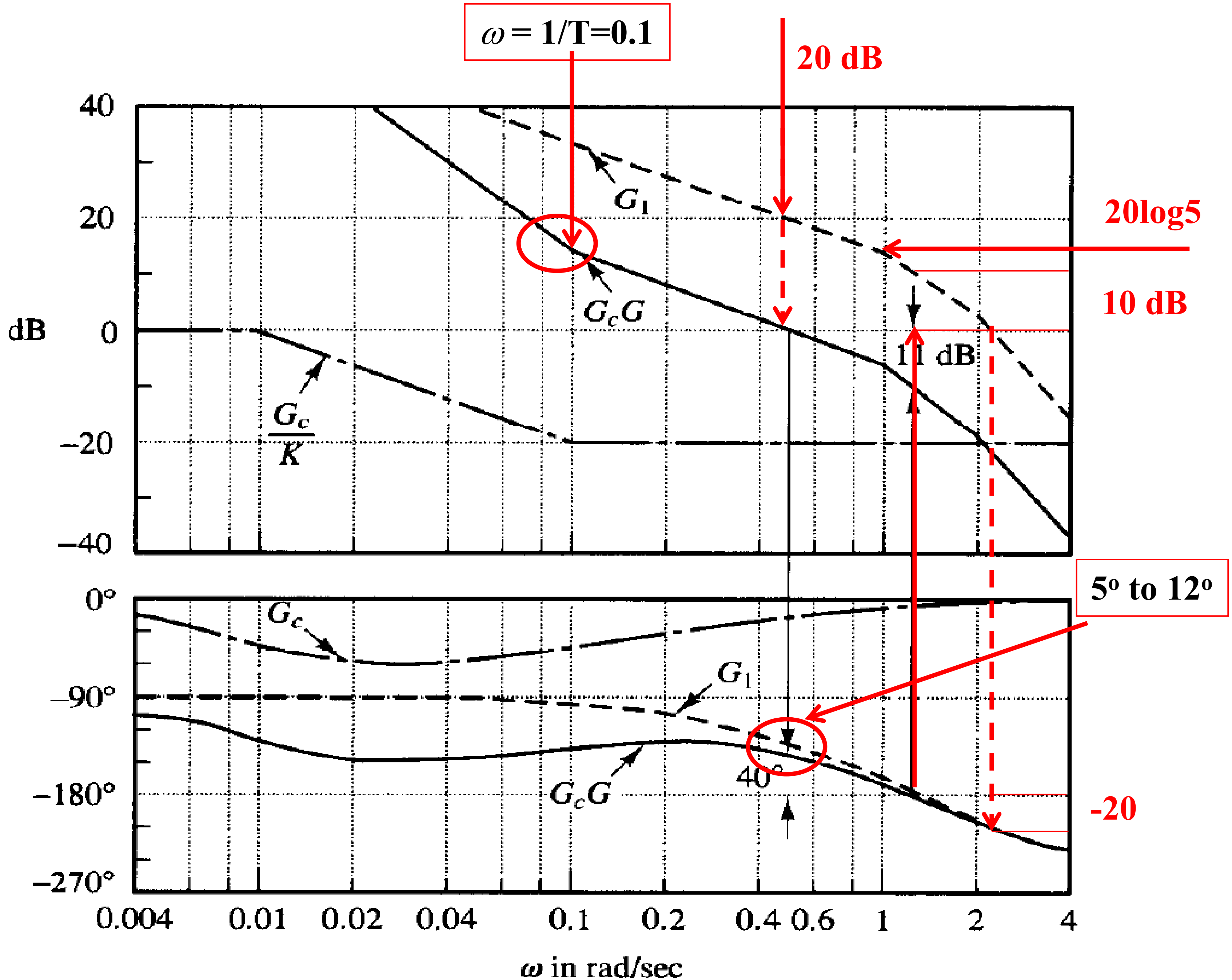
- **To meet the requirement,**

$$K_v = \lim_{s\to 0} sG_c(s)G(s) = \lim_{s\to 0} s\,\frac{Ts+1}{\beta Ts+1}\,G_1(s)$$

- **Therefore,** $K_v = \lim_{s\to 0} \dfrac{sK}{s(s+1)(0.5s+1)} = K = 5$

- **Next plot the Bode Diagram for**

$$G_1(j\omega) = KG(j\omega) = \frac{5}{j\omega(j\omega+1)(0.5j\omega+1)}$$



$\omega = 1/T = 0.1$
20 dB
20log5
10 dB
11 dB
G_1
G_cG
$\frac{G_c}{K}$
dB
40
20
0
–20
–40
5° to 12°
0°
–90°
–180°
–270°
G_c
G_1
G_cG
40°
-20
0.004
0.01
0.02
0.04
0.1
0.2
0.4
0.6
1
2
4
ω in rad/sec

- **The phase margin is found to be -20^o, therefore unstable.**
- **Chose the corner frequency ω = 1/T (which corresponds to the zero of the lag compensator) to be 0.1 rad/sec.**
- **The addition of a log compensator will modify the phase curve of the Bode Diagram, we must allow 5^o to 12^o to compensate the this shift.**
- **The requirement is 40^o, 40^o +12^o =52^o, therefore the phase angle is -128^o at about ω = 0.5 rad/sec.**

- **To bring the magnitude curve down to 0 dB at this new gain crossover frequency, the lag compensator must give the necessary attenuation, -20 dB at this point (ω = 0.5 rad/sec).**
- **Since** $AR = \left|G_1(j\omega)G_2(j\omega)\cdots\right| = \left|G_1(j\omega)\right|\left|G_2(j\omega)\right|\cdots$
- **and $K_c\beta = K$, therefore, to cancel out 20 dB of G_1 at (ω = 0.5 rad/sec), let**

$$20\log\frac{1}{\beta} = -20$$
$$\therefore \beta = 10$$

- **The other corner frequency $\omega = 1/(\beta T)$ is therefore determined as 0.01 rad/sec.**
- **Thus the transfer function of the lag compensator is,**

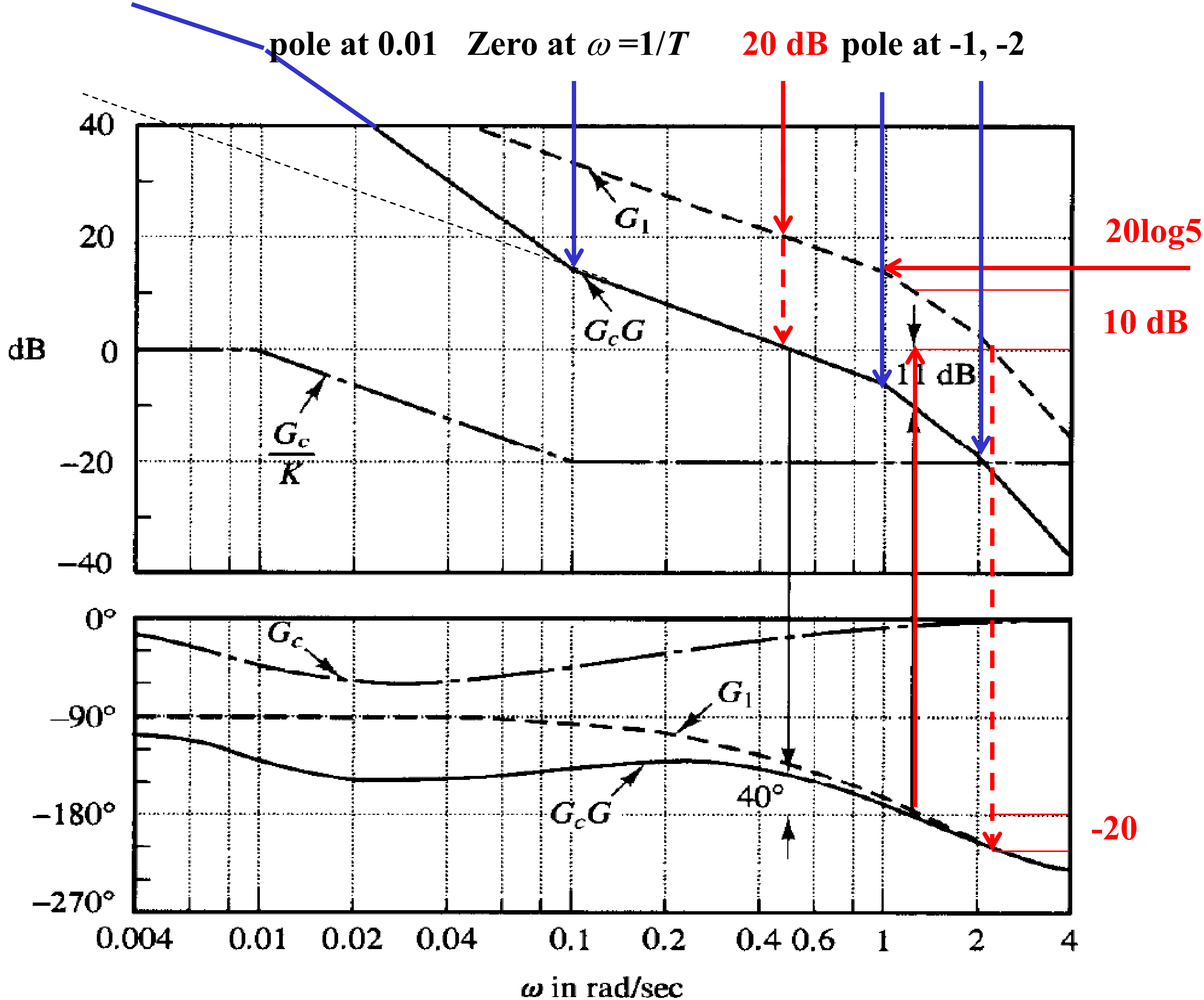
$$G_c(s) = K_c(10)\frac{10s+1}{100s+1} = K_c\frac{s+\frac{1}{10}}{s+\frac{1}{100}}$$

- **Since the gain K was determined to be 5 and β was determined to be 10,**
- $K_c\beta = K = 5$, $K_c = 0.5$

- **The open-loop transfer function of the compensated system is therefore,**

$$G_c(s)G(s) = \frac{5(10s+1)}{s(100s+1)(s+1)(0.5s+1)}$$

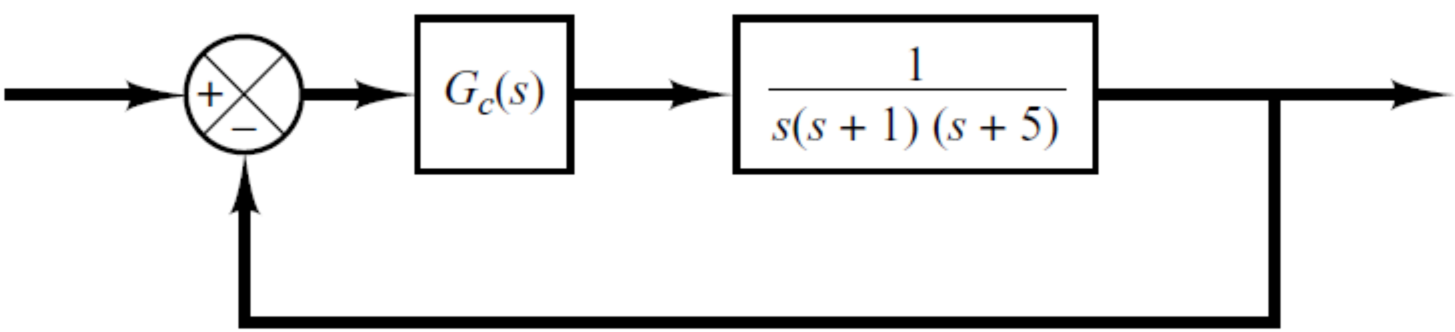
- **The phase margin is about 40^{o}, gain margin is about 11dB**



pole at 0.01
Zero at ω =1/T
20 dB
pole at -1, -2
20log5
10 dB
11 dB
-20
40
20
0
−20
−40
dB
G1
GcG
Gc/K
0°
−90°
−180°
−270°
Gc
G1
GcG
40°
0.004
0.01
0.02
0.04
0.1
0.2
0.4
0.6
1
2
4
ω in rad/sec

Lead-Lag

- Design a lag–lead compensator such that the static velocity error constant K_v is 50 sec^{-1} and the damping ratio ζ of the dominant closed loop poles is 0.5. (Choose the zero of the lead portion of the lag–lead compensator to cancel the pole at s=–1 of the plant.) Determine all closed-loop poles of the compensated system.



- **Choose,**

$$G_c(s) = K_C\left(\frac{s+\frac{1}{T_1}}{s+\frac{\beta}{T_1}}\right)\left(\frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}}\right) = K_C\,\frac{(T_1 s+1)(T_2 s+1)}{\left(\frac{T_1}{\beta}s+1\right)(\beta T_2 s+1)}$$

- **then**

$$K_V = \lim_{s\to 0} sG_c(s)G(s)$$

$$= \lim_{s\to 0} sK_C\,\frac{(T_1 s+1)(T_2 s+1)}{\left(\frac{T_1}{\beta}s+1\right)(\beta T_2 s+1)}\,\frac{1}{s(s+1)(s+5)}$$

$$= \frac{K_C}{5}$$

- K_v is 50 sec^{-1}, K_C =250.
- Let T_1 = 1 so that $(s+1/T_1)$ will cancel the (s+1) term of the plant, according to the statement
- The lead portion then becomes, $(s+1)/(s+\beta)$.
- The lag portion, hope to have

$$\left| \frac{s_1 + \frac{1}{T_2}}{s_1 + \frac{1}{\beta T_2}} \right| \cong 1, \qquad -5^\circ < \angle \frac{s_1 + \frac{1}{T_2}}{s_1 + \frac{1}{\beta T_2}} < 0^\circ$$

- $s = s_1$ is one of the dominant closed-loop poles.

- Noting these requirements for the lag portion of the compensator, at $s = s_1$, the open-loop transfer function becomes.

$$G_c(s_1)G(s_1) = K_C\left(\frac{s_1 + 1}{s_1 + \beta}\right)\frac{1}{s_1(s_1 + 1)(s_1 + 5)}$$

$$= K_C \frac{1}{s_1(s_1 + 1)(s_1 + 5)}$$

- Then at $s = s_1$, the following magnitude and angle conditions must be satisfied,

$$\left|K_C \frac{1}{s_1(s_1 + 1)(s_1 + 5)}\right| = 1$$

- and

$$\angle K_C \frac{1}{s_1(s_1+1)(s_1+5)} = \pm 180^\circ(2k+1)$$

- Notice β and s_1 are still unknown.
- Given ζ of the dominant closed-loop poles is specified as 0.5, assume (trial and error)

$$s_1 = -x + j\sqrt{3}x$$

- Substitute into the magnitude condition,

$$\left| \frac{K_C = 250}{s_1(s_1+1)(s_1+5)} \right| = 1$$

- Simplify we have,

$$x\sqrt{(\beta - x)^2 + 3x^2}\sqrt{(5-x)^2 + 3x^2} = 125$$

$$\frac{K_C = 250}{(-x + j\sqrt{3}x)(-x + \beta + j\sqrt{3}x)(-x + 5 + j\sqrt{3}x)}$$

$$= -120^{\circ} - \tan^{-1}\left(\frac{\sqrt{3}x}{-x + \beta}\right) - \tan^{-1}\left(\frac{\sqrt{3}x}{-x + 5}\right)$$

$$= -180^{\circ}$$

$$\tan^{-1}\left(\frac{\sqrt{3}x}{-x + \beta}\right) + \tan^{-1}\left(\frac{\sqrt{3}x}{-x + 5}\right) = 60^{\circ}$$

- We arrive at

$$\beta = 16.025, \qquad x = 1.9054$$

$$s_1 = -x + j\sqrt{3}x = -1.9054 + j3.3002$$

- Noting that the pole and zero of the lag portion of the compensator must be located near the origin, we may choose

$$\frac{1}{\beta T_2} = 0.01$$

- That is

$$\frac{1}{T_2} = 0.16025$$

- Substitute, $T_2 = 6.25$,

$$\left|\frac{s_1 + \frac{1}{T_2}}{s_1 + \frac{1}{\beta T_2}}\right| = \left|\frac{-1.9054 + j3.3002 + 0.16025}{-1.9054 + j3.3002 + 0.01}\right| = 0.98 \cong 1$$

- and

$$\angle \frac{s_1 + \frac{1}{T_2}}{s_1 + \frac{1}{\beta T_2}} = \angle \frac{-1.9054 + j3.3002 + 0.16025}{-1.9054 + j3.3002 + 0.01}$$

$$= \tan^{-1}\left(\frac{3.3002}{-1.74515}\right) - \tan^{-1}\left(\frac{3.3002}{-1.89054}\right) = -1.937^{\circ}$$

$$G_c(s) = 250\left(\frac{s + 1}{s + 16.025}\right)\left(\frac{s + 0.16025}{s + 0.01}\right)$$

$$\frac{C(s)}{R(s)} = \frac{250(s + 0.16025)}{s(s + 0.01)(s + 5) + 250(s + 16.025)}$$

- closed-loop poles are at

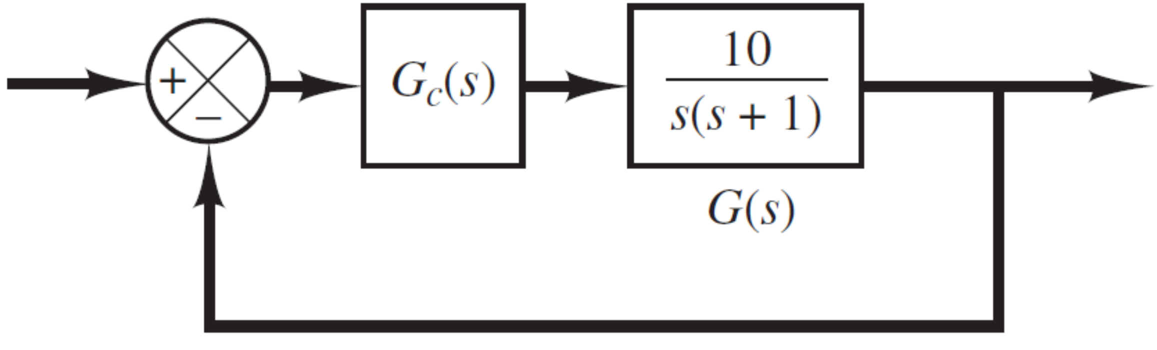
$$s = -1.8308 \pm j3.2359$$

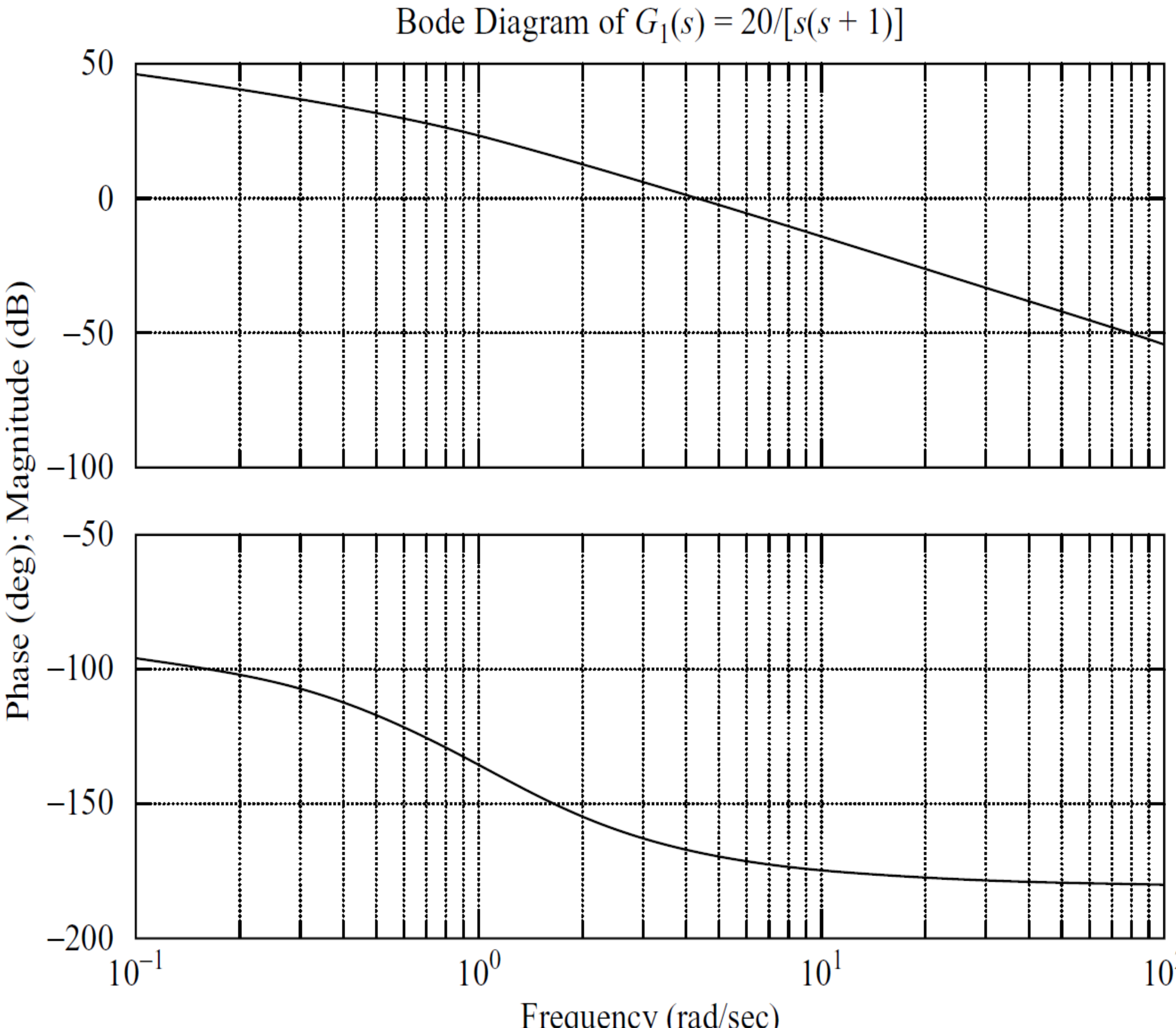
$$s = -0.1684$$

$$s = -17.205$$

Exercise

- Consider the system shown in Figure. Design a compensator such that the closed-loop system will satisfy the requirements that the static velocity error constant=20 sec^{-1}, phase margin=50°, and gain margin G 10 dB.




Bode Diagram of $G_1(s) = 20/[s(s+1)]$
50
0
−50
−100
−50
−100
−150
−200
Phase (deg); Magnitude (dB)
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)

- Since the specification calls for a phase margin of 50°, the additional phase lead necessary to
- satisfy the phase-margin requirement is 36°. A lead compensator can contribute this amount.
- Taking the shift of the gain crossover frequency into consideration, we may assume that fm, the maximum phase lead required, is approximately 41°.
- ϕm=41° corresponds to α=0.2077. Note that α=0.21 corresponds to ϕm=40.76°.
- let us choose α=0.21.

- The amount of the modification in the magnitude curve at $\omega = 1/\sqrt{\alpha}T$ due to the inclusion of the compensator, $(Ts+1)/(\alpha Ts+1)$, is

$$\left|\frac{1 + j\omega T}{1 + j\omega\alpha T}\right|_{\omega=\frac{1}{\sqrt{\alpha}T}} = \left|\frac{1 + j\dfrac{1}{\sqrt{\alpha}}}{1 + j\alpha\dfrac{1}{\sqrt{\alpha}}}\right| = \frac{1}{\sqrt{\alpha}}$$

$$\frac{1}{\sqrt{\alpha}} = \frac{1}{\sqrt{0.21}} = 6.7778 \text{ dB}$$

- From this diagram, we find the frequency point where $G_1(j\omega) = -6.7778$ dB occurs at $\omega = 6.5686$ rad/sec,

$$\omega_c = \frac{1}{\sqrt{\alpha}T}$$

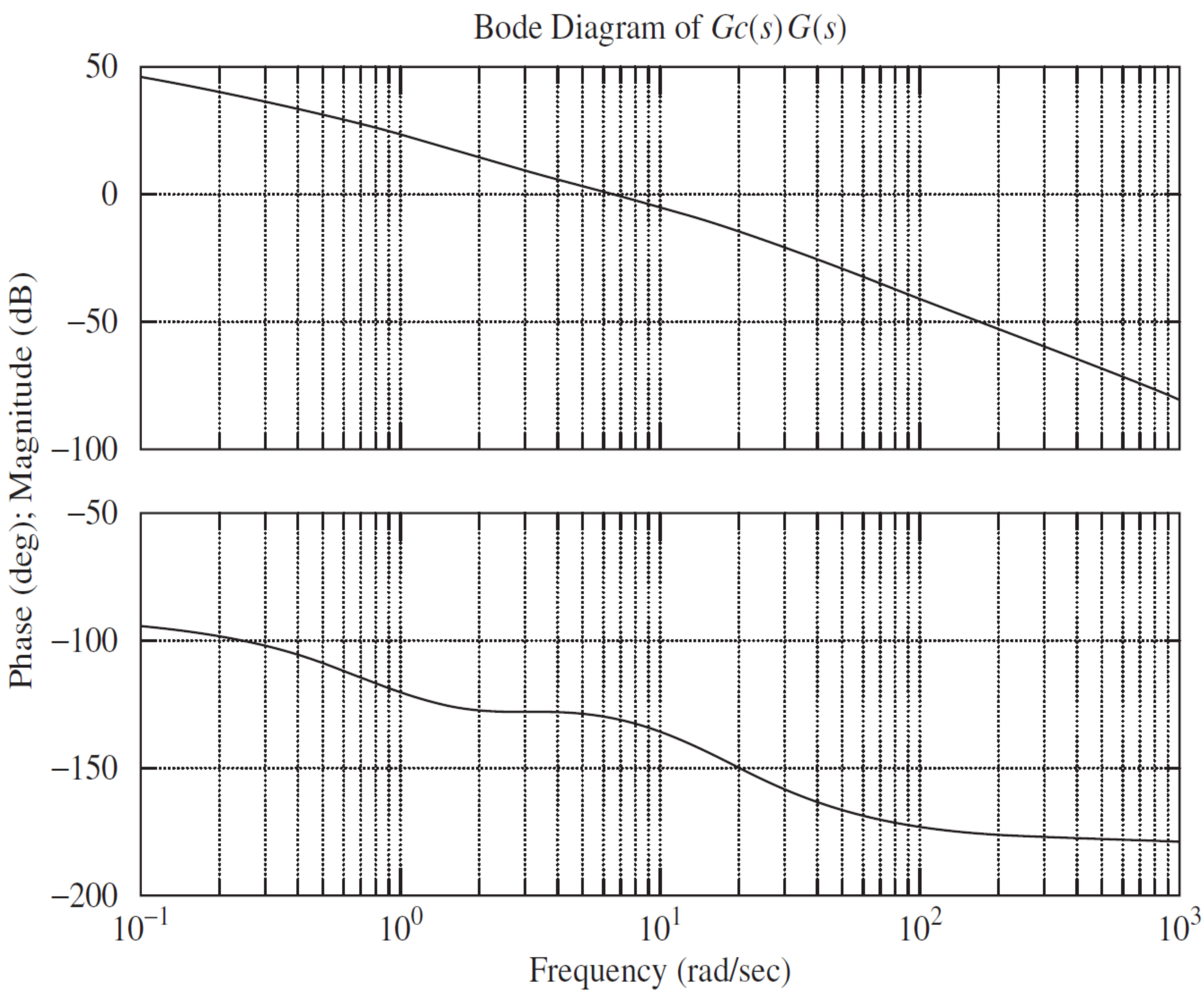
$$\frac{1}{T} = \omega_c\sqrt{\alpha} = 6.5686\sqrt{0.21} = 3.0101$$

$$\frac{1}{\alpha T} = \frac{\omega_c}{\sqrt{\alpha}} = \frac{6.5686}{\sqrt{0.21}} = 14.3339$$

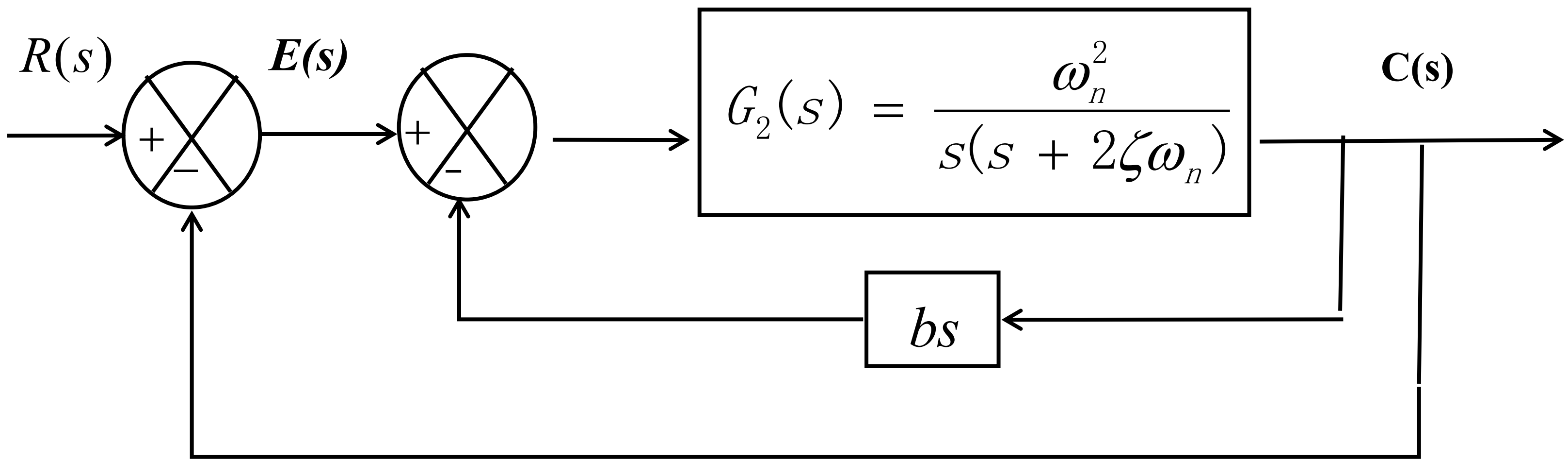
$$G_c = K_c \frac{s + 3.0101}{s + 14.3339} = K_c \alpha \frac{0.3322s + 1}{0.06976s + 1}$$

$$K_c = \frac{K}{\alpha} = \frac{2}{0.21} = 9.5238$$

$$G_c = 9.5238 \frac{s + 3.0101}{s + 14.3339} = 2 \frac{0.3322s + 1}{0.06976s + 1}$$


Bode Diagram of Gc(s) G(s)
Phase (deg); Magnitude (dB)
50
0
−50
−100
−50
−100
−150
−200
10^−1
10^0
10^1
10^2
10^3
Frequency (rad/sec)

Extra Homework



$$G(s) = \frac{16}{s(s+4)}$$

- b =0, determine the damping ratio undamped natural frequency, peak overshoot from a unit step input and the steady-state error resulting from a unit ramp input.
- Determine b which will increase the equivalent damping ratio of the system to 0.8
- The resulting steady-state error from unit ramp input.