Smith's Method

- **Assumed model:**

$$G(s) = \frac{Ke^{-\theta s}}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

If the original process transfer function contains a time delay

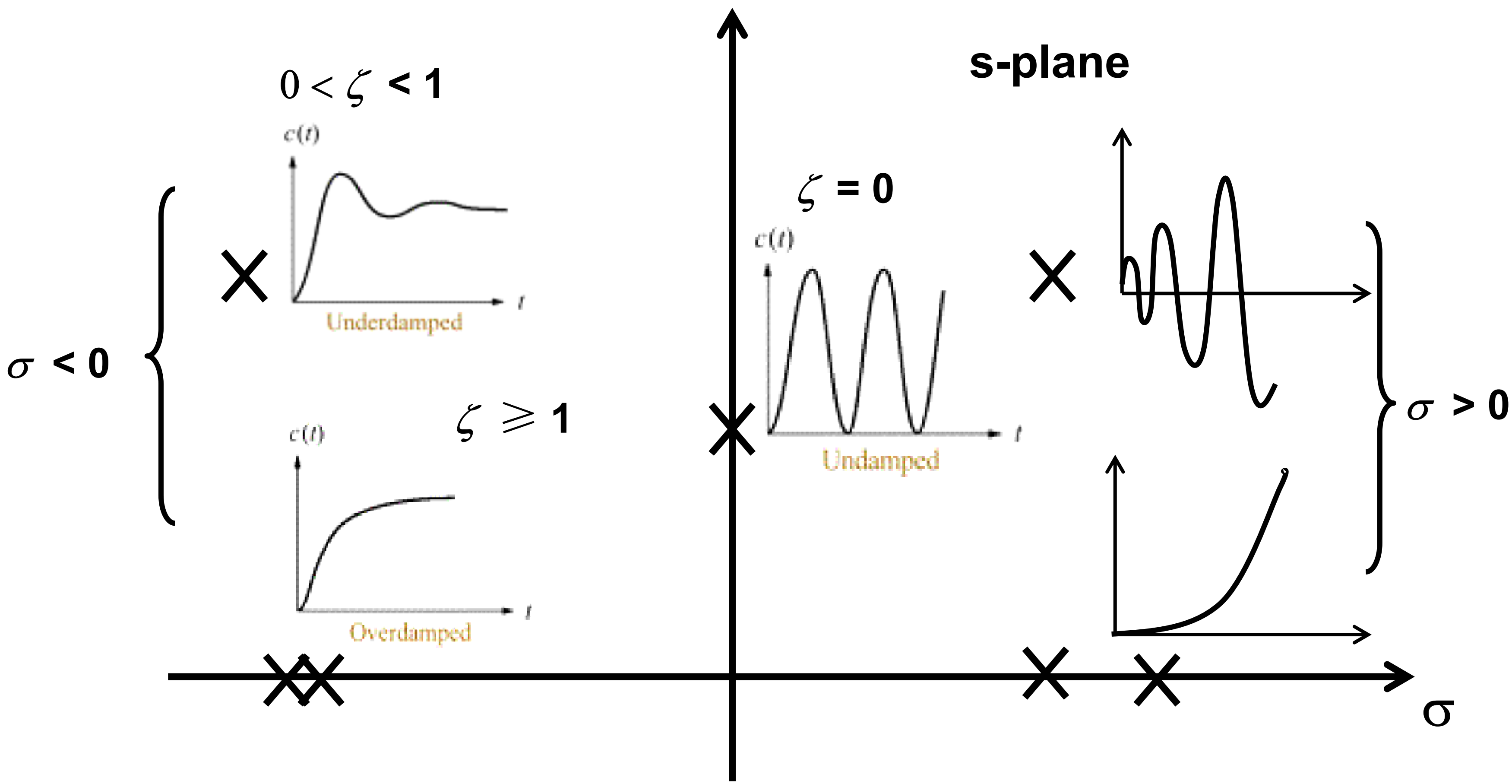
$$t = t' - \theta$$

Performance Criteria

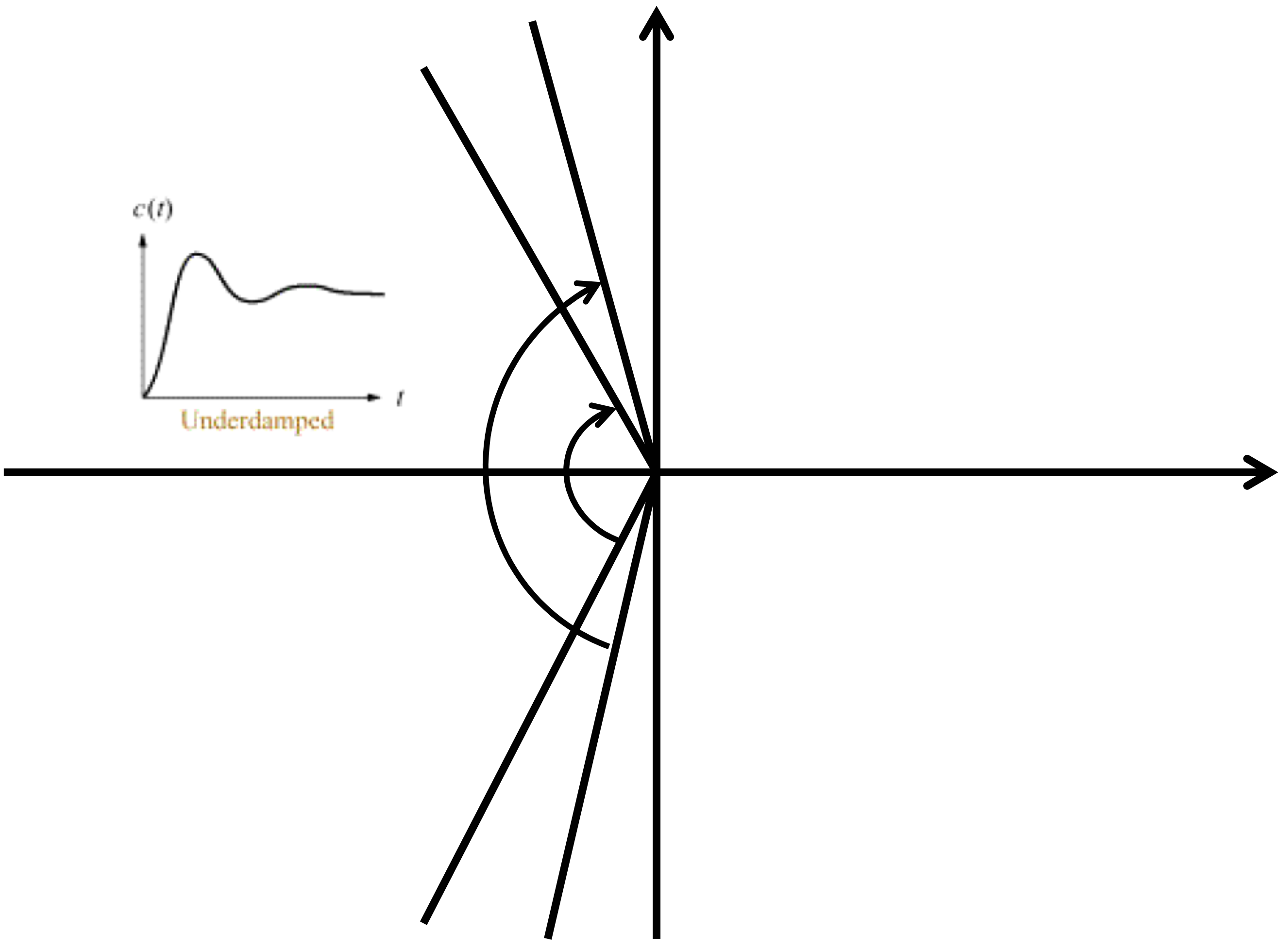
Performance of a Feedback Control System

- In the early days
  - Stability
  - Static Accuracy
- Modern Complex Control System Also Interested in
  - Sensitivity
  - Transient Response
  - Residual Noise Jitter

Stability


s-plane
$0 < \zeta$ < 1
c(t)
t
Underdamped
σ < 0
$\zeta \geqslant$ 1
c(t)
t
Overdamped
ζ = 0
c(t)
t
Undamped
σ > 0
σ

Stability



Sensitivity

- **Definition**

$$S_K^H(s) = \frac{d\ln H(s)}{d\ln K(s)} = \frac{\%\ \text{change in}\ H(s)}{\%\ \text{change in}\ K(s)}$$

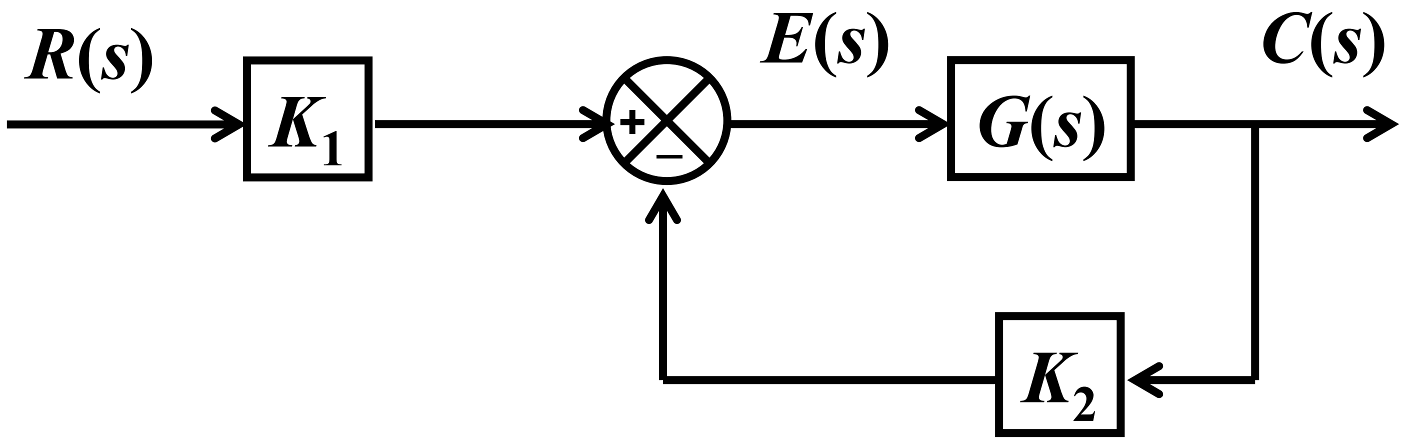
where

$$H(s) = \frac{C(s)}{R(s)}$$

- **Rearrange**

$$S_K^H(s) = \frac{dH(s)/H(s)}{dK(s)/K(s)}$$

Example



$$H(s) = \frac{K_1(s)G(s)}{1 + K_2(s)G(s)}$$

Example

- **Sensitivity of *H*(s) with respect to K_1**

$$S_{K_1}^{H}(s) = \frac{dH(s)/H(s)}{dK_1(s)/K_1(s)} = \frac{K_1(s)}{H(s)}\frac{dH(s)}{dK_1(s)}$$

where

$$\frac{dH(s)}{dK_1(s)} = \frac{G(s)}{1+K_2(s)G(s)} = \frac{H(s)}{K_1(s)}$$

Therefore,

$$S_{K_1}^{H}(s) = \frac{dH(s)/H(s)}{dK_1(s)/K_1(s)} = 1$$

Example

- **Sensitivity of *H*(s) with respect to K_2**

$$S_{K_2}^{H}(s) = \frac{dH(s)/H(s)}{dK_2(s)/K_2(s)} = \frac{K_2(s)}{H(s)}\frac{dH(s)}{dK_2(s)}$$

  – **where**

$$\frac{dH(s)}{dK_2(s)} = \frac{0 - K_1 G^2(s)}{\left[1 + K_2(s)G(s)\right]^2}$$

$$= \frac{-K_2(s)G^2(s)}{K_1(s)\left[1 + K_2(s)G(s)\right]^2}$$

Example

– **Therefore**

$$S_{K_2}^{H}(s) = \frac{K_2(s)}{H(s)} \frac{-K_1^2 G^2(s)}{K_1(s)\left[1 + K_2(s)G(s)\right]^2}$$

$$= \frac{-K_2(s)}{H(s)} \frac{H^2(s)}{K_1(s)} = \frac{-K_2(s)G(s)}{1 + K_2(s)G(s)}$$

– **When** $K_2 G(s) >> 1$

$$S_{K_2}^{H}(s) = \frac{-K_2(s)G(s)}{1 + K_2(s)G(s)} \approx -1$$

Example

- **Sensitivity of *H*(s) with respect to *G*(s)**

$$S_G^H(s) = \frac{dH(s)/H(s)}{dG(s)/G(s)} = \frac{G(s)}{H(s)}\frac{dH(s)}{dG(s)}$$

where

$$\frac{dH(s)}{dG(s)} = \frac{[1+K_2(s)G(s)]K_1 - K_1(s)G(s)K_2(s)}{[1+K_2(s)G(s)]^2}$$

$$= \frac{K_1(s)}{[1+K_2(s)G(s)]^2}$$

Example

- **Therefore**

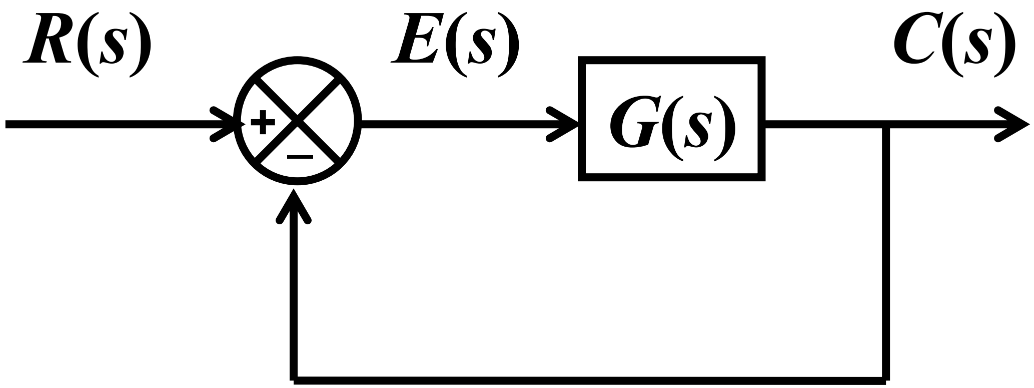
$$S_{G(s)}^{H}(s) = \frac{G(s)}{H(s)} \frac{K_1(s)}{\left[1 + K_2(s)G(s)\right]^2}$$

$$= \frac{1}{1 + K_2(s)G(s)}$$

- **Desire to have**

$$1 + K_2(s)G(s) >> 1$$

Static Accuracy



- **The transfer function,**

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)}$$

Static Accuracy

- **Apply the final-value theorem of the Laplace transform,**

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)}$$

- **We usually interested in,**
  - **Position input**
  - **Velocity input**
  - **Acceleration input**

Static Accuracy

- These inputs are usually called
  - Unit step

$$r(t) = U(t) \qquad R(s) = 1/s$$

  - Unit ramp

$$r(t) = tU(t) \qquad R(s) = 1/s^2$$

  - Paraboloid

$$r(t) = \frac{1}{2}t^2U(t) \qquad R(s) = 1/s^3$$

Static Accuracy

- **Assume the loop transfer function *G*(*s*),**

$$G(s) = \frac{K(1+T_1 s)(1+T_2 s)\cdots(1+T_m s)}{s^n\left[(T_0 s)^2 + 2\zeta\omega_n s + 1\right](1+T_a s)(1+T_b s)\cdots(1+T_q)}$$

- **Based on the pure integrations in the denominator of the open loop transfer function, the system is defined as the n^{th} type system**

Static Accuracy

- As the system type is increased, the accuracy is improved. However,
- in practice, we usually never design a control system greater than type 2 because it is very difficult to stabilize a control system containing more than two pure integrations.
- There will be a tradeoff between them

Static Accuracy

- **Unit step**

$$e_{ss} = \lim_{s \to 0} \frac{s(1/s)}{1+G(s)} = \frac{1}{1+\lim_{s \to 0} G(s)}$$

- **Define position constant**

$$K_p = \lim_{s \to 0} G(s)$$

- **therefore**

$$e_{ss} = \frac{1}{1+K_p}$$

Static Accuracy

- **Unit ramp**

$$e_{ss} = \lim_{s \to 0} \frac{s\left(1/s^2\right)}{1+G(s)} = \frac{1}{\lim_{s \to 0} sG(s)}$$

- **Define velocity constant**

$$K_v = \lim_{s \to 0} sG(s)$$

- **therefore**

$$e_{ss} = \frac{1}{K_v}$$

Static Accuracy

- **Parabolic**

$$e_{ss} = \lim_{s\to 0} \frac{s\left(1/s^3\right)}{1+G(s)} = \frac{1}{\lim_{s\to 0} s^2 G(s)}$$

- **Define acceleration constant**

$$K_a = \lim_{s\to 0} s^2 G(s)$$

- **therefore**

$$e_{ss} = \frac{1}{K_a}$$

Static Accuracy

- **If the input composed of all three forms,**

$$r(t) = AU(t) + BtU(t) + \frac{1}{2}Ct^2U(t)$$

  - **No pure integration**

$$e_{ss} = \frac{1}{1+K_p} + \infty + \infty$$

Static Accuracy

– **One pure integration**

$$e_{ss} = 0 + \frac{B}{K_v} + \infty$$

– **Two pure integration**

$$e_{ss} = 0 + 0 + \frac{C}{K_a}$$

Static Accuracy

- **Relationship between static error constants to closed-loop poles and zeros**
  - **A close loop transfer function,**

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

  - **The relation between the input and error,**

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)}$$

Static Accuracy

– **And**

$$E(s) = R(s) - C(s)$$

– **It is evident that**

$$\frac{C(s)}{R(s)} = 1 - \frac{E(s)}{R(s)}$$

– **Assume C(s)/R(s) is as**

$$\frac{C(s)}{R(s)} = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} = \frac{K\Pi_{j=1}^{m}(s+z_j)}{\Pi_{j=1}^{n}(s+p_j)}$$

Static Accuracy

– **And expend *E*(*s*)/*R*(*s*) as power series in *s***

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)} = \frac{1}{1+K_p} + \frac{1}{K_v}s + \frac{1}{K_a}s^2 + \cdots$$

- **Position Constant**
  - **Let *s* approach 0,**

$$\lim_{s\to 0}\frac{E(s)}{R(s)} = \lim_{s\to 0}\frac{1}{1+G(s)} = \frac{1}{1+K_p}$$

  - **therefore** $\frac{C(0)}{R(0)} = 1 - \frac{E(0)}{R(0)} = \frac{K_p}{1+K_p}$

Static Accuracy

– **Solving for K_p in terms of $C(s)/R(s)$**

$$K_p = \frac{C(0)/R(0)}{1 - C(0)/R(0)}$$

$$\frac{C(0)}{R(0)} = \frac{K(0+z_1)(0+z_2)\cdots(0+z_m)}{(0+p_1)(0+p_2)\cdots(0+p_n)} = \frac{K\Pi_{j=1}^{m} z_j}{\Pi_{j=1}^{n} p_j}$$

– **therefore**

$$K_p = \frac{K\Pi_{j=1}^{m} z_j}{\Pi_{j=1}^{n} p_j - K\Pi_{j=1}^{m} z_j}$$

Static Accuracy

- **Velocity Constant**

$$\frac{C(s)}{R(s)} = 1 - \frac{E(s)}{R(s)}$$

  - **And**

$$\frac{C(s)}{R(s)} = 1 - \frac{E(s)}{R(s)} = 1 - \left(\frac{1}{1+K_p} + \frac{1}{K_v} s + \frac{1}{K_a} s^2 + \cdots \right)$$

  - **Take derivative with respect to *s***

$$\frac{d}{ds}\left(\frac{C(s)}{R(s)} \right) = -\frac{1}{K_v}$$

Static Accuracy

– **Therefore,**

$$\frac{1}{K_v} = -\frac{\left[\dfrac{d}{ds}\left(\dfrac{C(s)}{R(s)}\right)\right]_{s=0}}{\left[\dfrac{C(s)}{R(s)}\right]_{s=0}} = -\left[\frac{d}{ds}\ln\frac{C(s)}{R(s)}\right]_{s=0}$$

– **and**

$$\frac{1}{K_v} = -\left\{\frac{d}{ds}\left[\ln K + \ln(s+z_1) + \cdots + \ln(s+z_m) - \ln K - \ln(s+p_1) - \cdots - \ln(s+p_n)\right]\right\}_{s=0}$$

Static Accuracy

– **It may be simplified as**

$$\frac{1}{K_v} = -\left(\frac{1}{s+z_1} + \cdots + \frac{1}{s+z_m} - \frac{1}{s+p_1} - \cdots - \frac{1}{s+p_n} \right)_{s=0}$$

– **or**

$$\frac{1}{K_v} = \sum_{j=1}^{n} \frac{1}{p_j} - \sum_{j=1}^{m} \frac{1}{z_j}$$

Static Accuracy

- **Acceleration Constant**

$$\frac{d^2}{ds^2}\left[\ln\frac{C(s)}{R(s)}\right]=\frac{\frac{d^2}{ds^2}\left[\frac{C(s)}{R(s)}\right]}{\frac{C(s)}{R(s)}}-\left\{\frac{\frac{d}{ds}\left[\frac{C(s)}{R(s)}\right]}{\frac{C(s)}{R(s)}}\right\}^2$$

  - **recall**

$$\frac{C(s)}{R(s)}=1-\frac{E(s)}{R(s)}=1-\left(\frac{1}{1+K_p}+\frac{1}{K_v}s+\frac{1}{K_a}s^2+\cdots\right)$$

Static Accuracy

– **Therefore**

$$\left\{\frac{d^2}{ds^2}\left[\ln\frac{C(s)}{R(s)}\right]\right\}_{s=0} = -\frac{1}{K_v^2} - \frac{2}{K_a}$$

– **Recall**

$$-\left[\frac{d}{ds}\ln\frac{C(s)}{R(s)}\right]_{s=0} = \frac{1}{K_v} = \sum_{j=1}^{n}\frac{1}{p_j} - \sum_{j=1}^{m}\frac{1}{z_j}$$

Static Accuracy

– **Therefore**

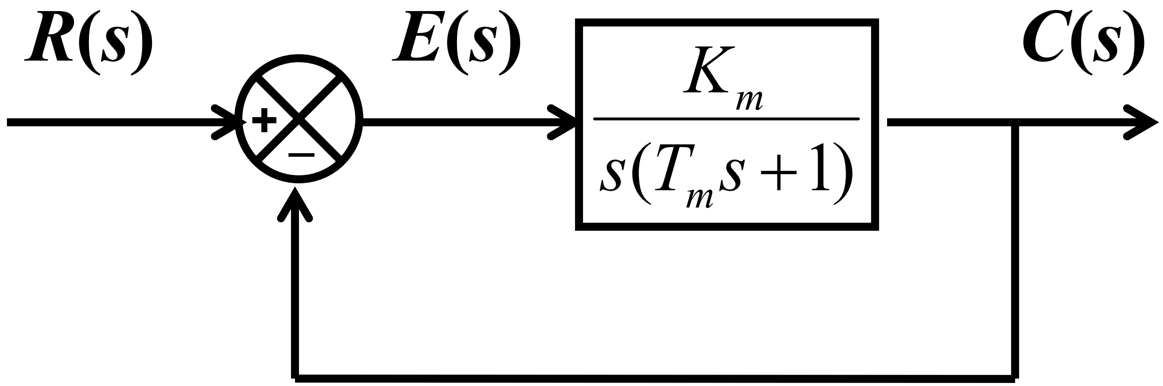
$$\sum_{j=1}^{n}\frac{1}{p_j^2}-\sum_{j=1}^{m}\frac{1}{z_j^2}=-\frac{1}{K_v^2}-\frac{2}{K_a}$$

- **Example**

$$\frac{C(s)}{R(s)}=\frac{K_m/T_m}{s^2+\left(1/T_m\right)s+K_m/T_m}$$

Where K_m is the system gain and T_m is the time constant of the open-loop transfer function

Static Accuracy



- **Or in the more familiar form**

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Static Accuracy

- **From the definition,**

$$K_v = \lim_{s \to 0} sG(s) = \lim_{s \to 0} s \frac{K_m}{s(T_m s + 1)} = K_m$$

- **therefore,**

$$K_v = K_m = \frac{\omega_n}{2\zeta}$$

This is of great importance, to have very accurate response to velocity response, ζ must be very small—in inertial navigation applications, such as missile control.

Zero-Error System

- **General form,**

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$= \frac{A_1 s^n + A_2 s^{n-1} + \cdots A_n s + A_{n+1}}{B_1 s^m + B_2 s^{m-1} + \cdots B_m s + B_{m+1}}$$

- **Recall**

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)} = \frac{1}{1+K_p} + \frac{1}{K_v}s + \frac{1}{K_a}s^2 + \cdots$$

Zero-Error System

- **Therefore**

$$e_{ss} = \frac{r(t)}{1+K_p} + \frac{\dot{r}(t)}{K_v} + \frac{\ddot{r}(t)}{K_a} + \cdots$$

- **It has been proved that for e_{ss} = 0, with a step input,**

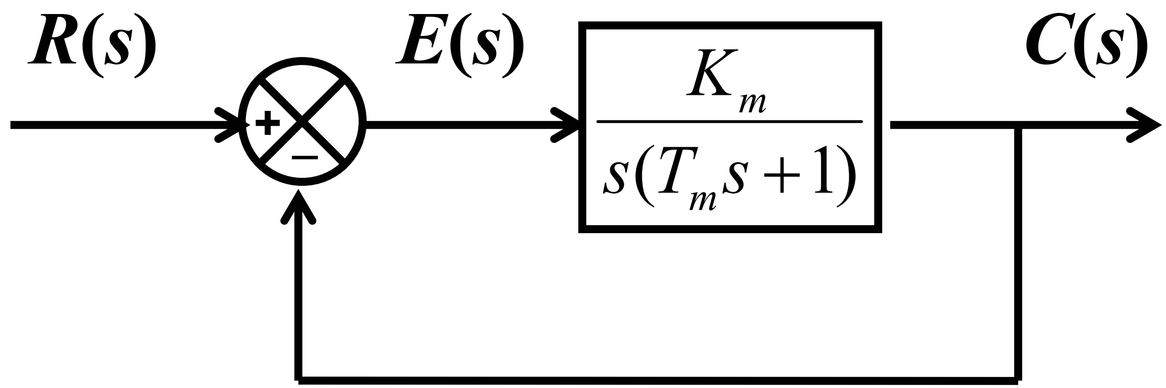
$\frac{1}{1+K_p}$ **is a function of B_{m+1} - A_{n+1}, and $B_{m+1} = A_{n+1}$**

Zero-Error System

- **If,** $$\frac{C(s)}{R(s)} = \frac{B_{m+1}}{B_1 s^m + B_2 s^{m-1} + \cdots B_m s + B_{m+1}}$$

  The zero steady-state step error system

- **Example (one pure integrator)**



Zero-Error System

- **Example**

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$= \frac{\dfrac{K_m}{s(T_m s+1)}}{1+\dfrac{K_m}{s(T_m s+1)}} = \frac{K_m}{T_m s^2 + s + K_m}$$

A zero steady-state step error system (prove it)

Zero-Error System

- It has also been proved that for e_{ss} = 0, with a ramp input,

$\frac{1}{K_v}$ is a function of $B_{m+1} - A_{n+1}$, $B_m - A_n$, and

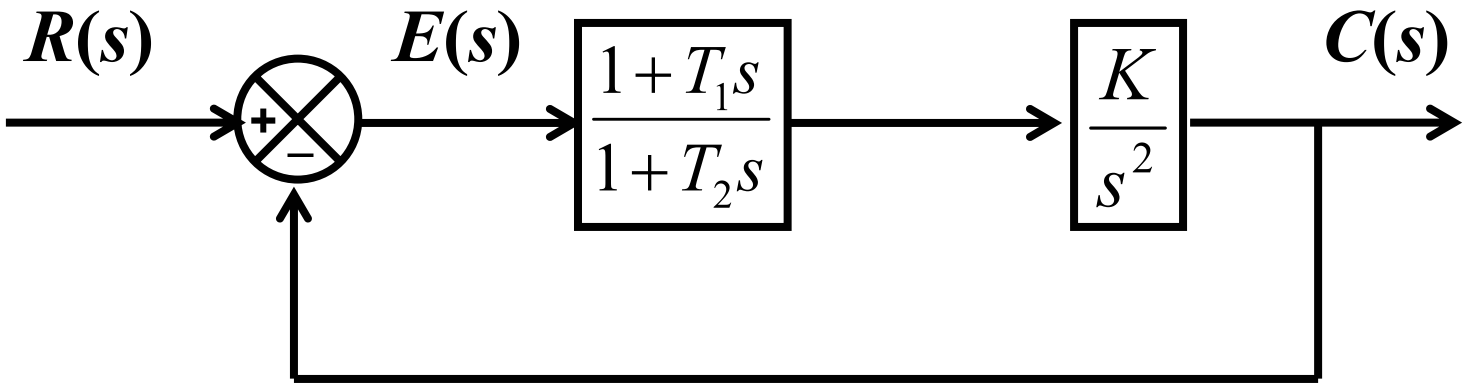
$B_{m+1} = A_{n+1}$ and $B_m = A_n$,

Zero-Error System

- **It can be shown that for ramp input,**

$$\frac{C(s)}{R(s)} = \frac{B_m s + B_{m+1}}{B_1 s^m + B_2 s^{m-1} + \cdots B_m s + B_{m+1}}$$

is a zero steady-state ramp error system



Zero-Error System

- **It contains two pure integrates**

$$\frac{C(s)}{R(s)} = \frac{\dfrac{T_1 s + 1}{T_2 s + 1}\dfrac{K}{s^2}}{1 + \dfrac{T_1 s + 1}{T_2 s + 1}\dfrac{K}{s^2}} = \frac{K\left(T_1 s + 1\right)}{T_2 s^2 + s^2 + K T_1 s + K}$$

- **It is a zero steady-state ramp error system**