Stability

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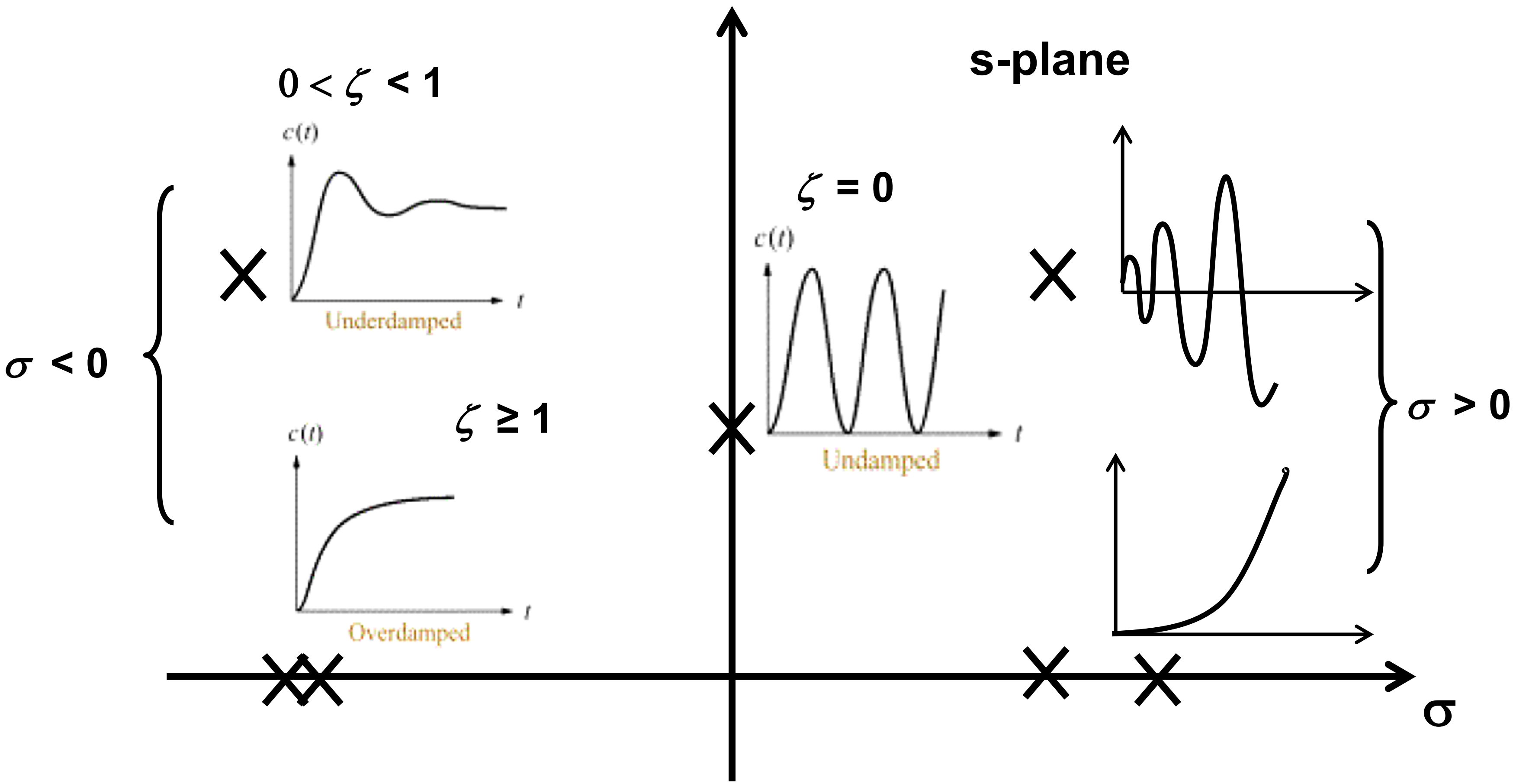
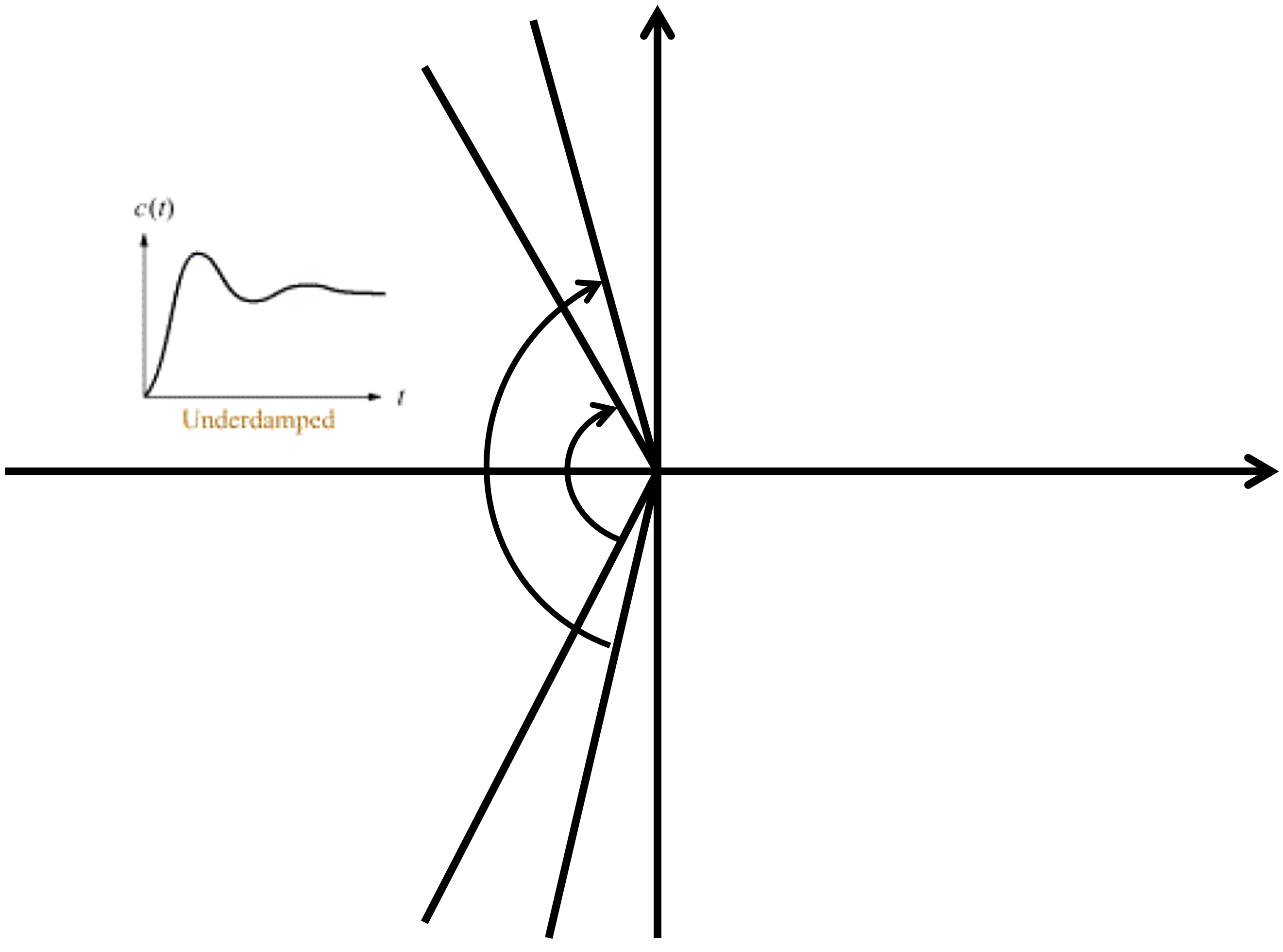


s-plane
$0 < \zeta < 1$
c(t)
t
Underdamped
$\zeta \geq 1$
c(t)
t
Overdamped
$\zeta = 0$
c(t)
t
Undamped
$\sigma < 0$
$\sigma > 0$
σ

Stability



Routh-Hurwitz Stability Criterion

- **Assume the characteristic polynomial is**

$$1 + G(s)H(s) = B_1 s^n + B_2 s^{n-1} + \cdots + B_n s + B_{n+1}$$

- **where** $B_{\text{n+1}} \neq 0$
- **A necessary (but not sufficient) condition for all roots to have non-positive real parts is that all coefficients have the same sign.**
- **All coefficients must be nonzero.**

The Routh Array

$$Q(s) = B_1 s^n + B_2 s^{n-1} + \cdots + B_n s + B_{n+1}$$

$$\begin{array}{llllll}
s^n & B_1 & B_3 & B_5 & B_7 & \cdots \\
s^{n-1} & B_2 & B_4 & B_6 & B_8 & \cdots \\
s^{n-2} & U_1 & U_3 & U_5 & U_6 & \cdots \\
s^{n-3} & U_2 & U_4 & U_5 & U_8 & \cdots \\
\vdots & \vdots & \vdots & \vdots & & \\
s^2 & & & & & \\
s^1 & & & & & \\
s^0 & Z_1 & & & &
\end{array}$$

where

$$U_1 = \frac{B_2 B_3 - B_1 B_4}{B_2}$$

$$U_2 = \frac{U_1 B_4 - B_2 U_3}{U_1}$$

etc.

Routh-Hurwitz Stability Criterion

- Necessary and sufficient conditions:
- If all elements in the first column of the Routh array have the same sign, then all roots of the characteristic equation have negative real parts.
- If there are sign changes in these elements, then the number of roots with non-negative real parts is equal to the number of sign changes.
- Elements in the first column which are zero define a special case.

Routh-Hurwitz Stability Criterion

- **Consider** $1+G(s)H(s)=s^3+4s^2+100s+500=0$

$$
\begin{array}{c|cc}
S^3 & 1 & 100 \\
S^2 & 4 & 500 \\
S^1 & -25 & 0 \\
S^0 & 500 & 0
\end{array}
$$

- **unstable**

Routh-Hurwitz Stability Criterion

- **Consider** $1+G(s)H(s)=s^5+s^4+4s^3+4s^2+2s+1=0$

s^5		1	4	2
s^4		1	4	1
s^3	*Replace this* $\rightarrow$	$(0$	1	$0)$
	with $\rightarrow$	ε	1	0
s^2		$\frac{4\varepsilon-1}{\varepsilon}$	1	0
s		$\frac{-\varepsilon^2+4\varepsilon-1}{4\varepsilon-1}$	0	0
s^0		1	0	0

- **As ε approaches zero, 4th goes negative, 5th positive, unstable**

Routh-Hurwitz Stability Criterion

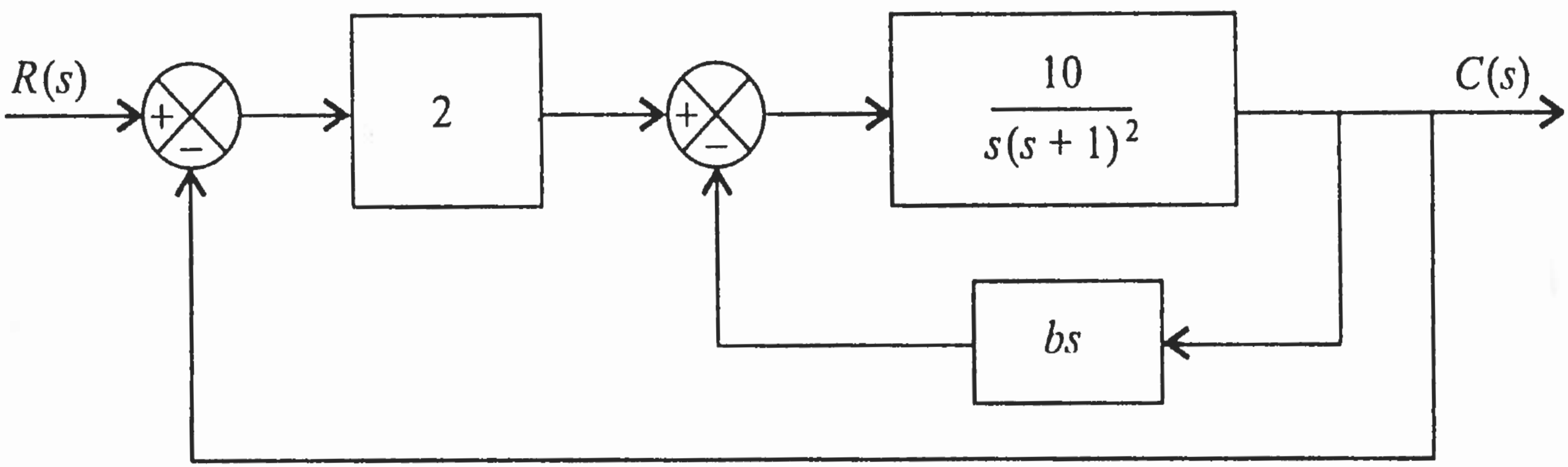
- **Consider** $1+G(s)H(s)=s^3+2s^2+s+K=0$

$$
\begin{array}{c|cc}
S^3 & 1 & 1 \\
S^2 & 2 & K \\
S^1 & U_1=\frac{2-K}{2} & \\
S^0 & K &
\end{array}
$$

- **0< K$_{max}$ = 2**

QUIZ 10min

A control system containing a tachometer, which provides rate feedback, is illustrated. Determine the range of the tachometer constant, b. in order that the system is always stable.



Nyquist Stability Criterion

- **It is based on the complex analysis result known as *Cauchy's principle of argument***
- **The system transfer function is a complex function**
- **(Nyquist, 1932), by applying Cauchy's principle of argument to the *open-loop system* transfer function, we will get information about stability of the closed-loop system transfer function and arrive at the Nyquist stability criterion**

Nyquist Stability Criterion

- **The importance of Nyquist stability lies in the fact that it can also be used to determine the relative degree of system stability by producing the so-called phase and gain stability margins. These stability margins are needed for frequency domain controller design techniques.**
- **The Nyquist method is used for studying the stability of linear systems with pure time delay.**

Nyquist Stability Criterion

- For a single input single output (SISO) feedback system the closed-loop transfer function is given by

$$L(s) = \frac{G(s)}{1 + G(s)H(s)}$$

- The closed-loop system poles are obtained by solving the following equation

$$1 + H(s)G(s) = 0 = \Delta(s)$$

Nyquist Stability Criterion

- Consider the complex function

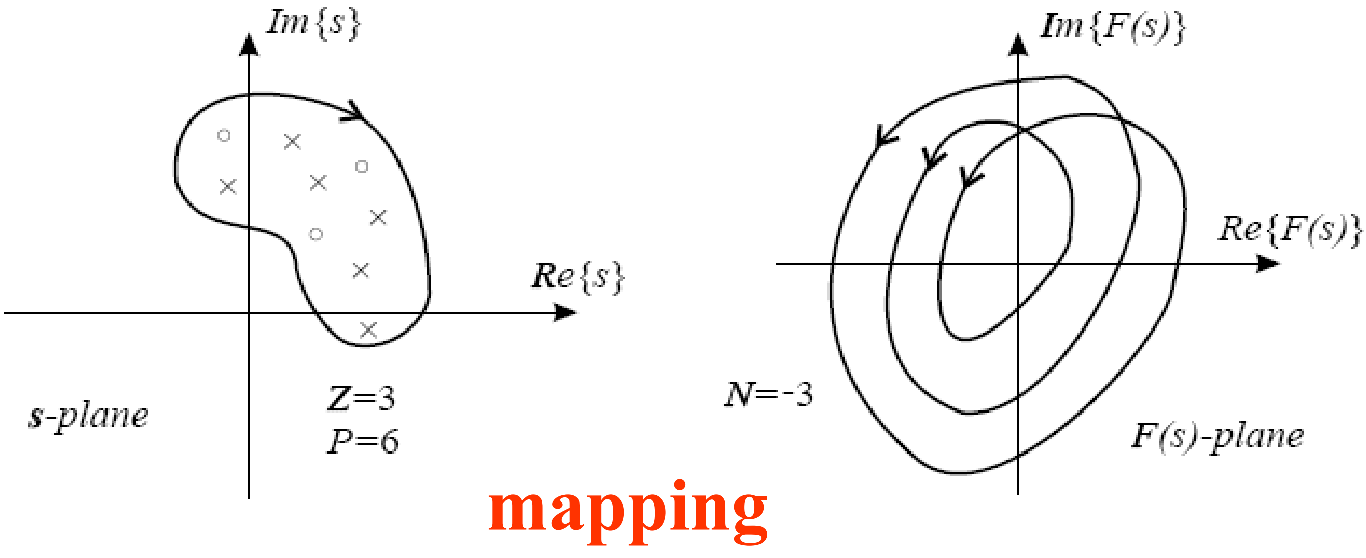
$$\Delta(s) = 1 + H(s)G(s)$$

  whose zeros are the closed-loop poles of the transfer function
- In addition, it is easy to see that the poles of Δ(s) are the zeros of **L**(s).
- At the same time the poles of Δ(s) are the open-loop control system poles.

$$L(s) = \frac{H(s)G(s)}{1 + H(s)G(s)} = H(s)G(s)\left(\Delta'(s)\right)^{-1}$$

Cauchy's principle of argument

- Let F(s) be an analytic function in a closed region of the complex plane s given in the Figure 1 except at a finite number of points (namely, the poles of F(s)).



Cauchy's principle of argument

- as s travels around the contour in the s-plane in the clockwise direction, the function encircles the origin in the $(Re\{F(s)\}, Im\{F(s)\})$-plane in the same direction N times, see the Figure, with N given by $N = Z - P$

  where z and p stand for the number of zeros and poles (including their multiplicities) of the function F(s) inside the contour

Cauchy’s principle of argument

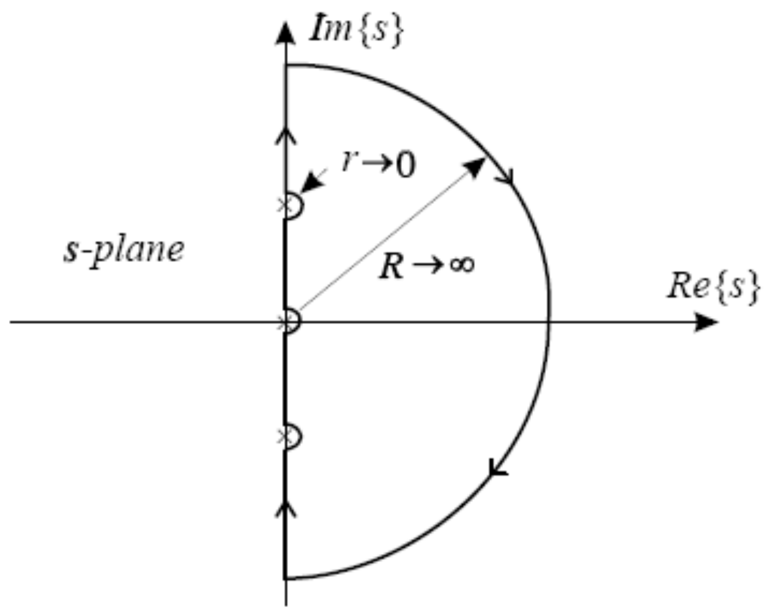
- The above result can be also written as

$$\arg\{F(s)\} = (Z - P)2\pi = 2\pi N$$

- which justifies the terminology used, “the principle of argument”.

Nyquist Plot

- **The Nyquist plot is a polar plot of the function** $D(s) = 1 + G(s)H(s)$
- **when travels around the contour given in the Figure 2**



Nyquist Plot

- **The contour in this figure covers the whole unstable half plane of the complex plane s, $R \rightarrow \infty$. Since the function D(s), according to Cauchy's principle of argument, must be analytic at every point on the contour, the poles of D(s) on the imaginary axis must be encircled by infinitesimally small semicircles**

Nyquist Stability Criterion

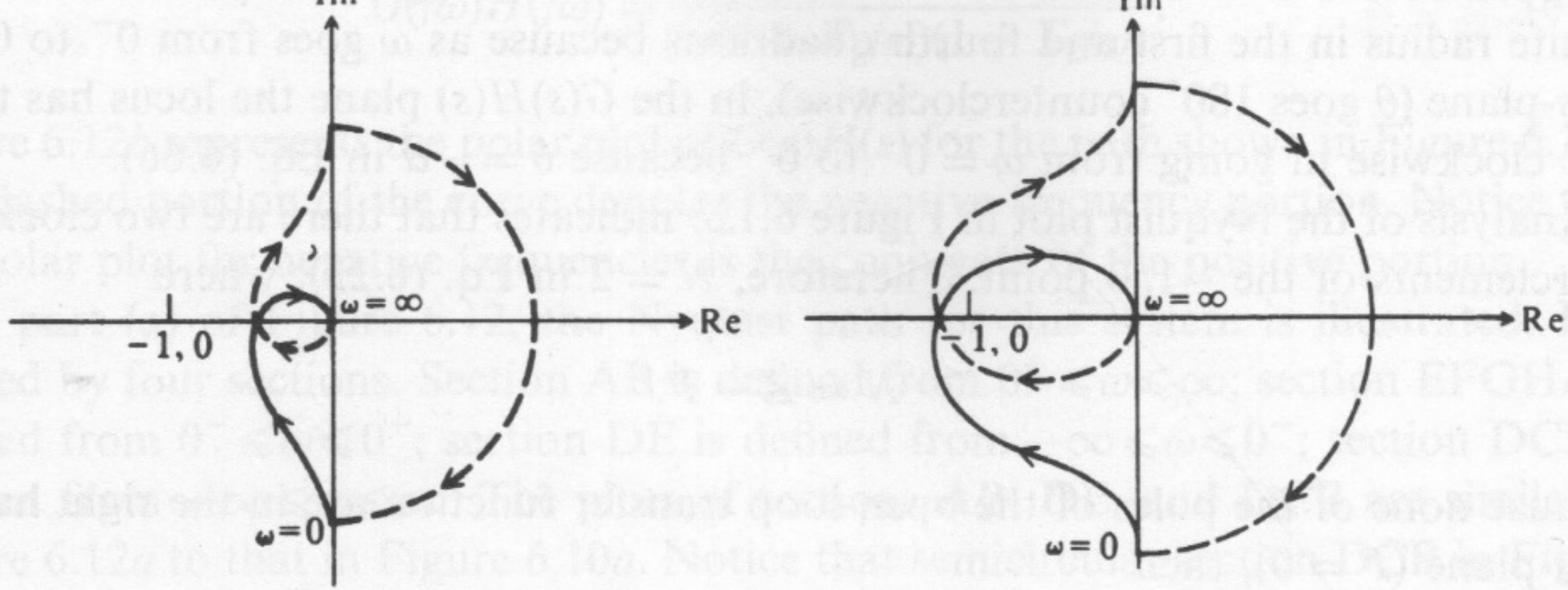
- The above criterion can be slightly simplified if instead of plotting the function $D(s) = 1 + G(s)H(s)$
- plot only the function G(s)H(s) and count encirclement (clock-wise) of the Nyquist plot of around the point (-1, j0), so that the modified Nyquist criterion has the following form $Z = P + N$

Nyquist Stability Criterion

- The number of unstable closed-loop poles (Z) is equal to the number of unstable open-loop poles (P) plus the number of encirclements (N) of the point (-1, j0), of the Nyquist plot of G(s)H(s)
- If the system is originally open-loop unstable Right-half-plane (RHP) poles represent that instability. For closed-loop stability of a system, the number of closed-loop roots in the right half of the s-plane must be zero.

Nyquist Stability Criterion

- **Hence, the number of counter-clockwise encirclements (-*N*) about (-1, j0), must be equal to the number of open-loop poles in the RHP.**
- **Any clockwise encirclements of the critical point by the open-loop frequency response (when judged from low frequency to high frequency) would indicate that the feedback control system would be destabilizing if the loop were closed.**

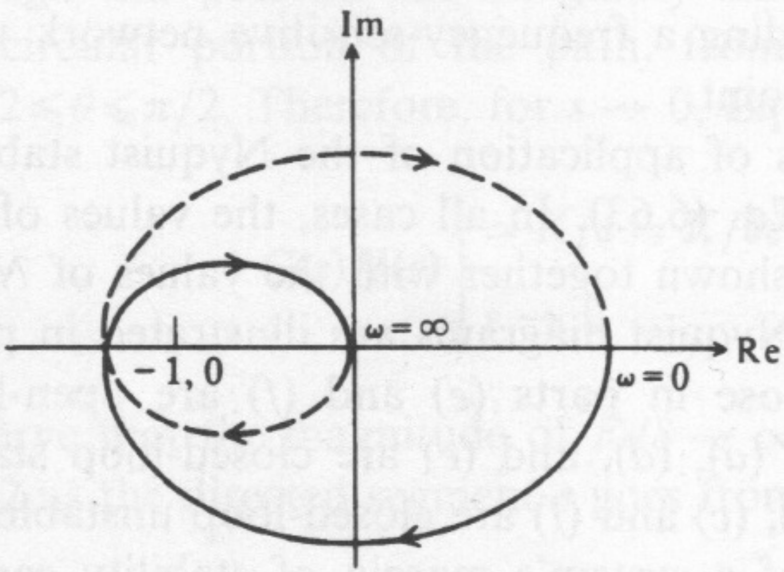


(a) $G(s)\,H(s) = \dfrac{K_a}{s(1+T_1 s)(1+T_2 s)}$

$N = 0$
$P = 0$
$Z = 0$; therefore stable

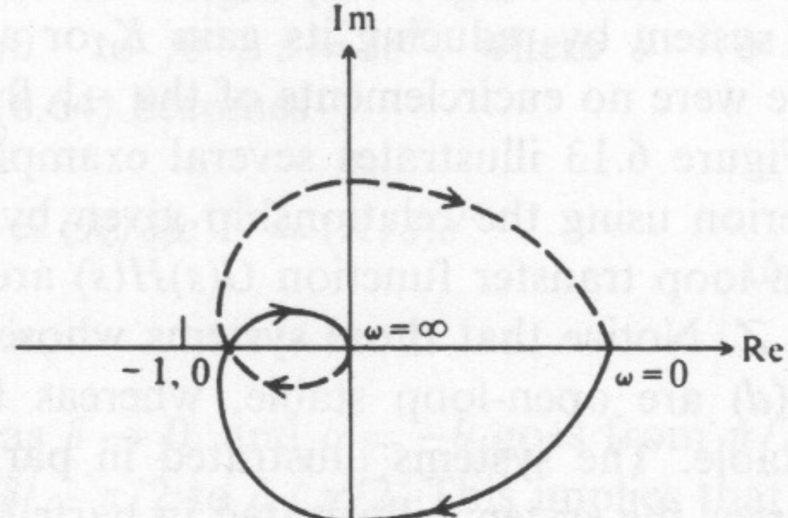
(b) $G(s)\,H(s) = \dfrac{K_b}{s(1+T_1 s)(1+T_2 s)}$

$N = 2$
$P = 0$
$Z = 2$; therefore unstable



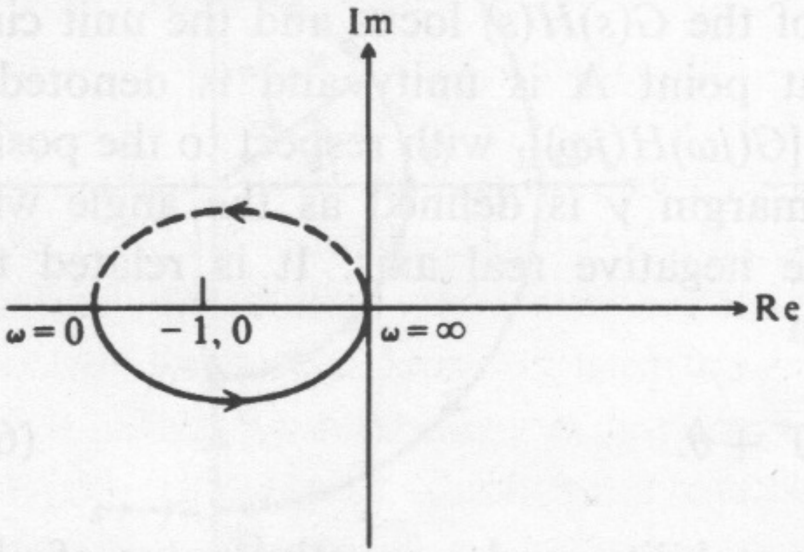
(c) $G(s)\,H(s) = \dfrac{K_c}{(1+T_1 s)\,(1+T_2 s)\,(1+T_3 s)}$

$N = 2$
$P = 0$
$Z = 2$; therefore unstable



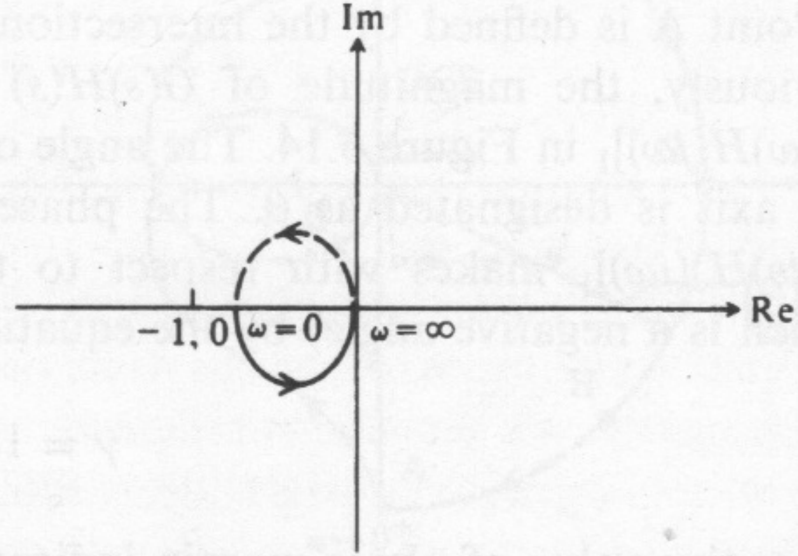
(d) $G(s)\,H(s) = \dfrac{K_d}{(1+T_1 s)\,(1+T_2 s)\,(1+T_3 s)}$

$N = 0$
$P = 0$
$Z = 0$; therefore stable



(e) $G(s)H(s) = \dfrac{K_e}{(-1 + Ts)}$

$N = -1$
$P = 1$
$Z = 0$; therefore stable

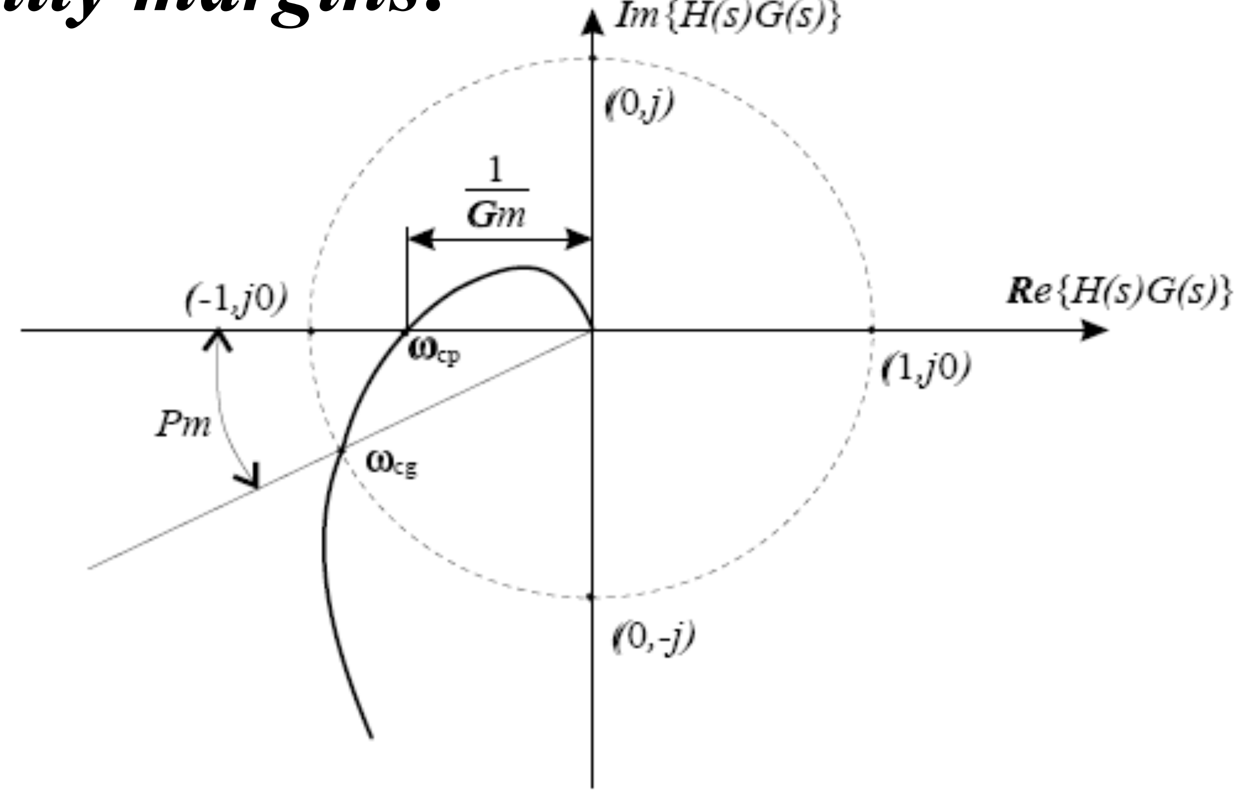


(f) $G(s)\,H(s) = \dfrac{K_f}{(-1 + Ts)}$

$N = 0$
$P = 1$
$Z = 1$; therefore unstable

Phase and Gain Stability Margins

- Two important notions can be derived from the Nyquist diagram: ***phase and gain stability margins***.



Phase and Gain Stability Margins

- **Give the degree of relative stability; in other words, they tell how far the given system is from the instability region. Their formal definitions are given by**

$$P_m = 180^o + \arg\{G(j\omega_{cg})H(j\omega_{cg})\}$$

$$G_m[dB] = 20\log\frac{1}{|G(j\omega_{cp})H(j\omega_{cp})|}[dB]$$

- **where ω_{cg} and ω_{cp} stand for, respectively, the gain and phase crossover frequencies,**

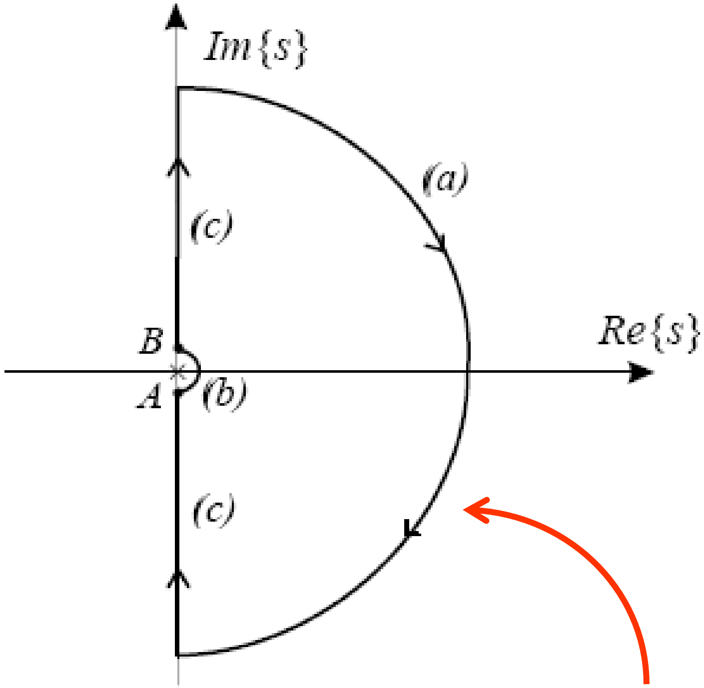
$$\arg\{G(j\omega_{cp})H(j\omega_{cp})\} = 180^o \Rightarrow \omega_{cp}$$

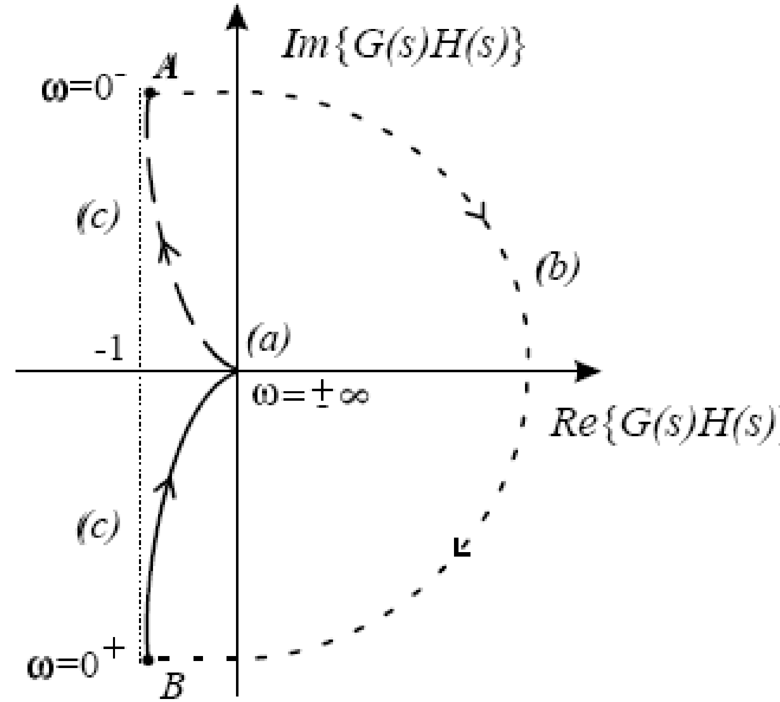
$$|G(j\omega_{cg})H(j\omega_{cg})| = 1 \Rightarrow \omega_{cg}$$

- **Example 1**

$$G(s)H(s) = \frac{1}{s(s+1)}$$

- **Since this system has a pole at the origin, the contour in the s-plane should encircle it with a semicircle of an infinitesimally small radius. This contour has three parts (a), (b), and (c). Mappings for each of them are considered below.**

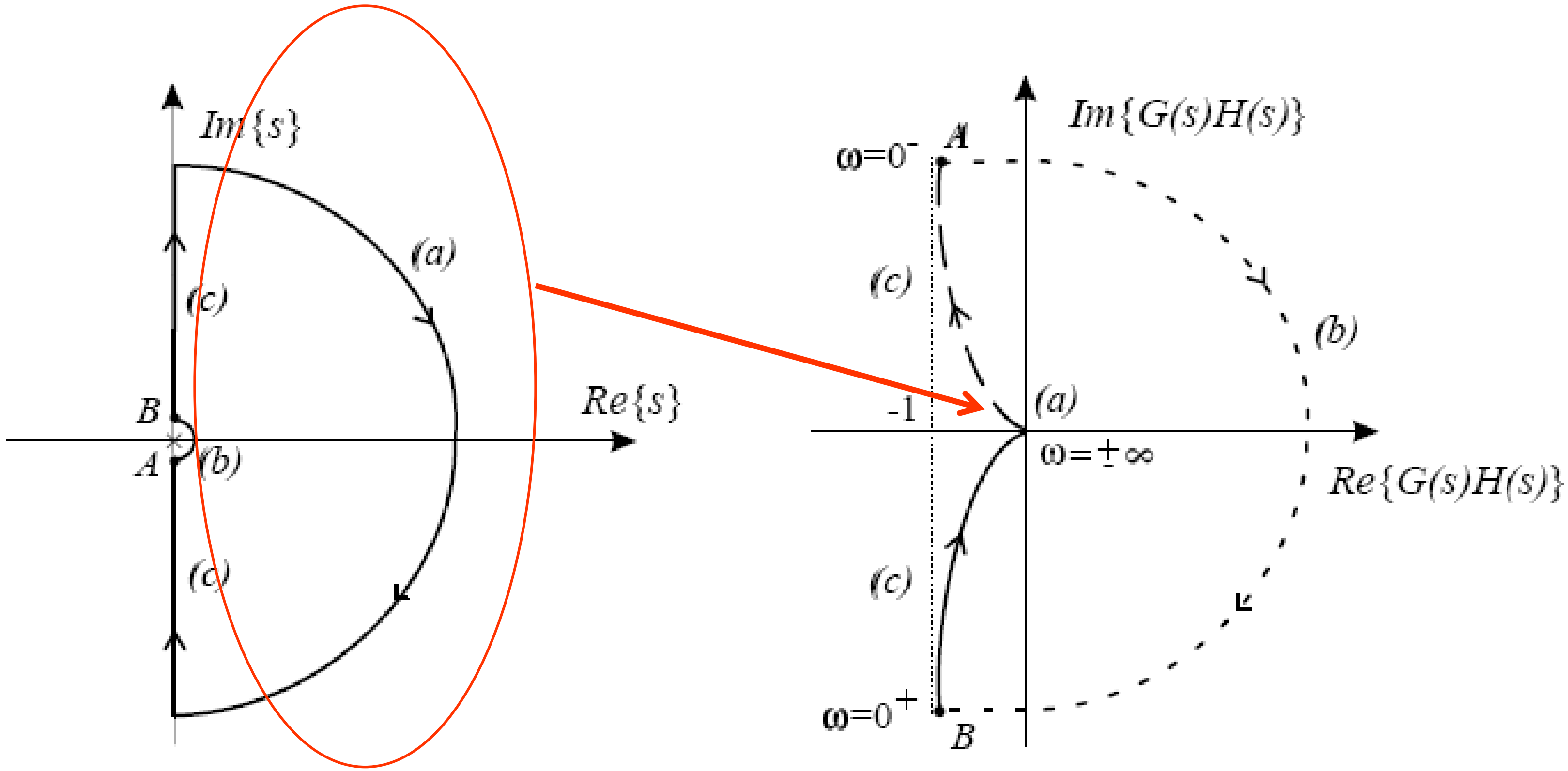




- **(a) On this semicircle the complex variable is represented in the polar form by**

$$s = Re^{j\Psi} \text{ with } R \to \infty, \ -\frac{\pi}{2} \le \Psi \le \frac{\pi}{2}$$

- **Substituting** $s = Re^{j\Psi}$ into $G(s)H(s)$
- **Then** $G(s)H(s) = \dfrac{1}{s(s+1)} \to 0$



- **Thus, the huge semicircle from the s-plane maps into the origin in the G(s)H(s)-plane**

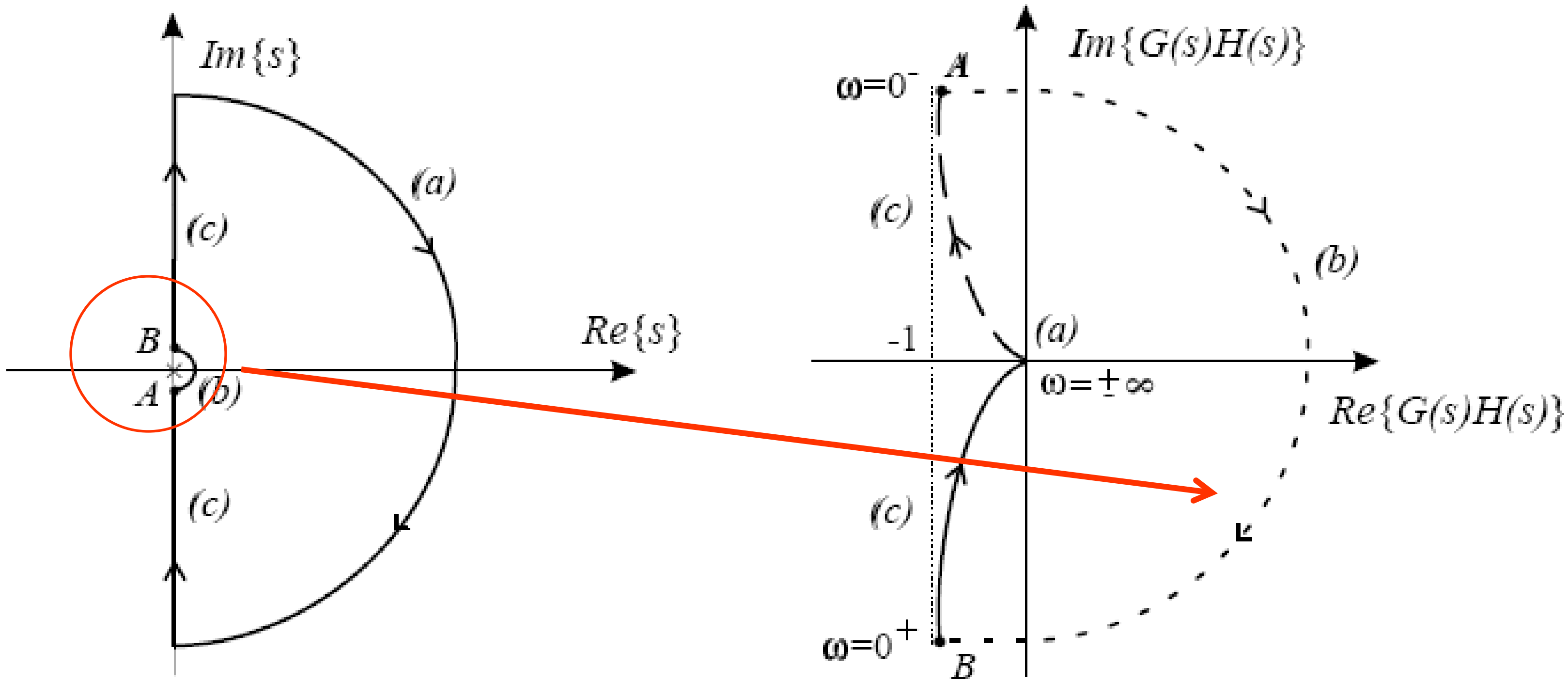
- **(b) On this semicircle the complex variable is represented in the polar form by**
- $s = re^{j\Phi}$ with $r \to 0$, $-\frac{\pi}{2} \le \Phi \le \frac{\pi}{2}$ **so that we**
- **have**

$$G(s)H(s) = \frac{1}{s(s+1)} \to \pm\infty$$

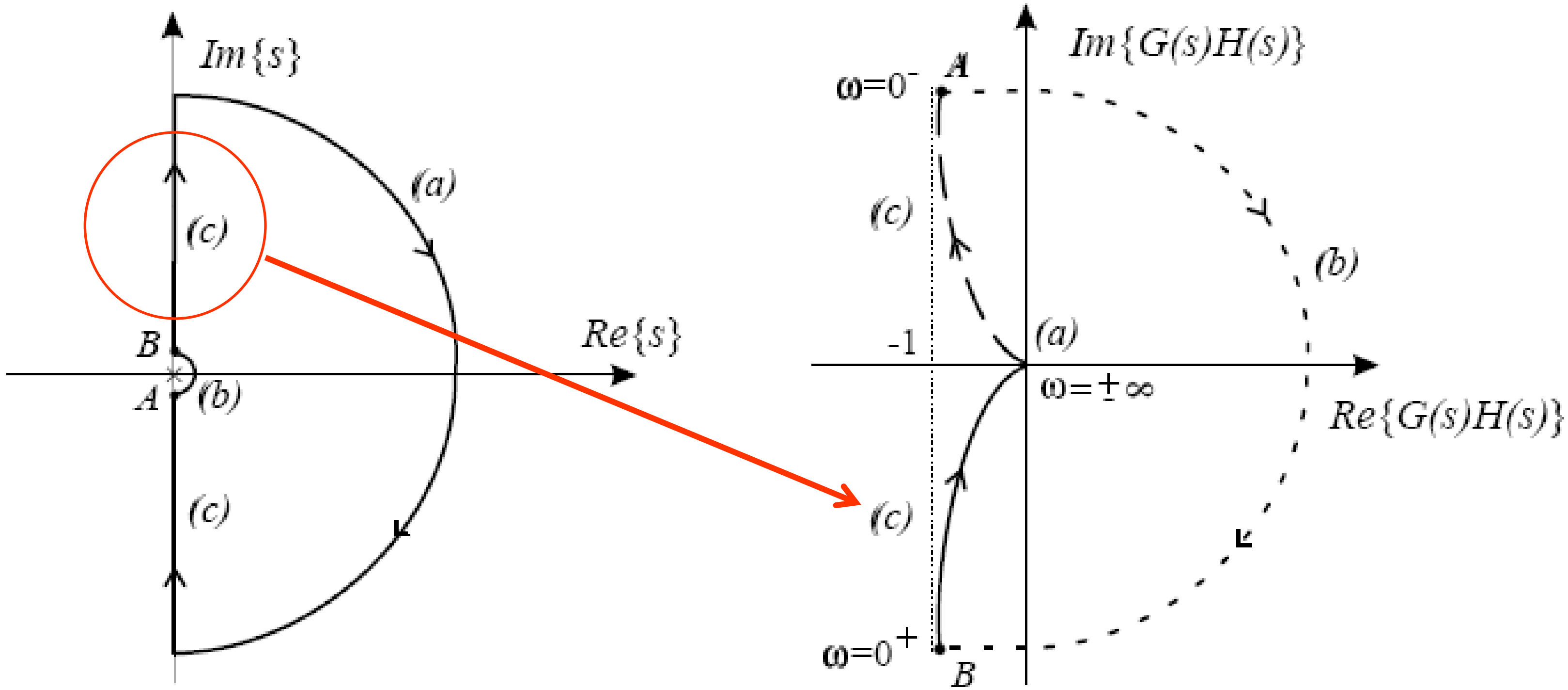
- **Since Φ changes from**

$-\frac{\pi}{2}$ at point A to $\frac{\pi}{2}$ at point B,

- $\arg\{G(s)H(s)\}$ **will change from** $\frac{\pi}{2}$ to $-\frac{\pi}{2}$



- **The infinitesimally small semicircle at the origin in the s-plane is mapped into a semicircle of infinite radius in the G(s)H(s)-plane.**

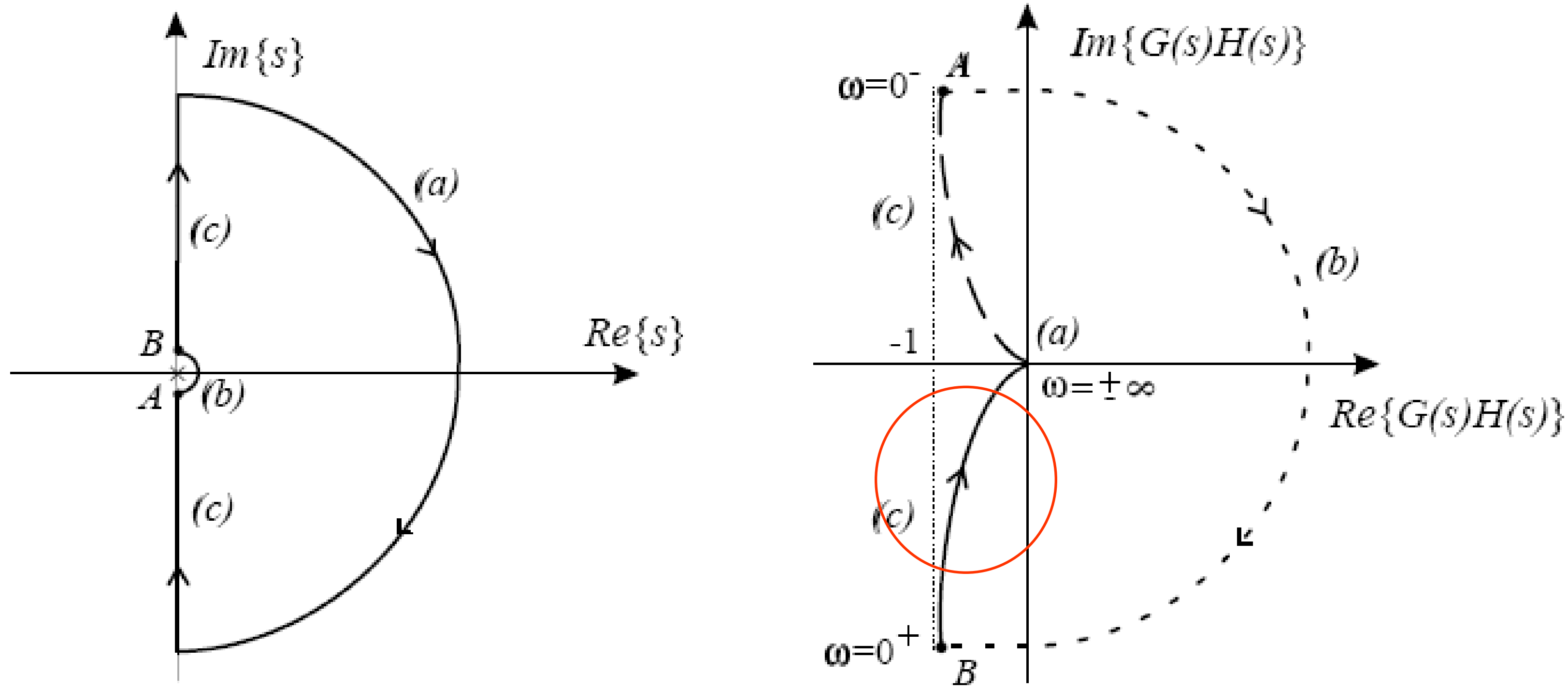


- imaginary values, i.e. s = jω with ω changing from $-\infty$ to $+\infty$ Due to symmetry, it is sufficient to study only mapping along $0^+ \leq \omega \leq +\infty$, the real and imaginary parts of the function G(jω)H(jω), which are given by

$$Re\{G(j\omega)H(jw)\} = \frac{-1}{\omega^2+1}$$

$$Im\{G(j\omega)H(jw)\} = \frac{-1}{\omega(\omega^2+1)}$$

- **neither the real nor the imaginary parts can be made zero, and hence the Nyquist plot has no points of intersection with the coordinate axis.**



- **The Nyquist diagram corresponding to part (c) has the form as shown in Figure.**
- **Note that the vertical asymptote 渐近线 of the Nyquist plot in Figure is given by**

$$Re\{G(j0^{\pm})H(j0^{\pm})\} = -1$$

since at those points $Im\{G(j0^{\pm})H(j0^{\pm})\} = \mp\infty$

- **From the Nyquist diagram we see that N = 0 and since there are no open-loop poles in the left half of the complex plane, i.e. p = 0, we have Z = 0 so that the corresponding closed-loop system has no unstable poles.**

- **Example 2**

$$G(s)H(s) = \frac{1}{s(s+1)(s+2)}$$

- **For cases (a) and (b) we have the same analyses and conclusions.**
- **the real and imaginary parts of G(jω)H(jω)**

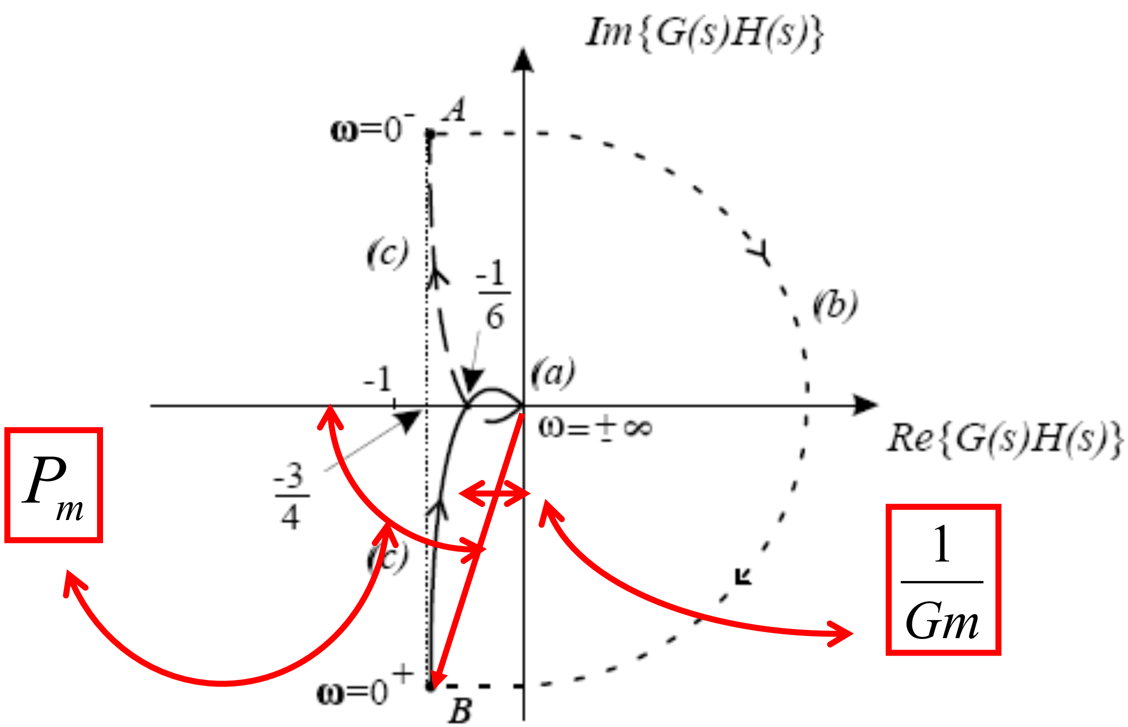
$$Re\{G(j\omega)H(j\omega)\} = \frac{-3}{9\omega^2 + (2-\omega^2)^2}$$

$$Im\{G(j\omega)H(j\omega)\} = \frac{-(2-\omega^2)}{\omega\left[9\omega^2 + (2-\omega^2)^2\right]}$$

- **It can be seen that an intersection with the real axis happens at $\omega = \sqrt{2}$ at the point**

$$R_e\left\{G\left(j\sqrt{2}\right)H\left(j\sqrt{2}\right)\right\} = -1/6$$

- **The Nyquist plot is given in Figure**



- **Note that the vertical asymptote is given by**

$$Re\{G(j0)H(j0)\} = -3/4$$

- **Thus, we have N = 0, P = 0, and Z = 0 and so that the closed loop system is stable**

$$Gm = 6\,\mathrm{dB}, \quad Pm = 53.4108^\circ$$

$$\omega_{cg} = 0.4457\,\mathrm{rad/s}, \quad \omega_{cp} = 1.4142\,\mathrm{rad/s}$$

QUIZ 15min

The open-loop transfer function of a feedback control system is given by the following:

$$G(s)H(s) = \frac{20}{s(0.2s + 1)(0.4s + 1)}.$$