

Double Interpolation

Two (independent) properties fix the state of a simple, pure substance. How many times have I made this point?

Sometime in the future (hopefully near future) students of thermodynamics will rely exclusively on computerized thermodynamic property software to implement the two-property principle. Until then, the property tables will be the primary source of property information, and to use these tables the student must be proficient in interpolation methods.

Here is an example of a double interpolation problem:

Find P for water at $T = 323^\circ\text{C}$, $v = 0.2 \text{ m}^3/\text{kg}$: This is a superheated state since $v > v_g(323^\circ\text{C})$. The temperature lies between listings for 300 and 350°C in the superheated tables. By inspection, we arrive at the following data points:

P , kPa	T , °C	v , m ³ /kg
1200	300	0.2134
1200	350	0.2345
1400	300	0.1823
1400	350	0.2003

Note that by ‘eyeballing’ the numbers, we can see that the volume for the first pair, at 323°C, will be somewhat larger than 0.2 m³/kg, and the volume for the second pair will be smaller. Our sought data point will therefore be bracketed by the four listed points.

Now interpolate using the first pair of data points to get v for 1200 kPa and 323°C. The general formula is

$$y = y_1 + (y_2 - y_1) \frac{x - x_1}{x_2 - x_1}$$

in which x is what we know, y is what we’re after, and subscripts 1 and 2 denote data point values. In general, x should be between x_1 and x_2 (if it is not, then we are *extrapolating*). The interpolation yields

P , kPa	T , °C	v , m ³ /kg
1200	323	0.2231
1400	323	0.1906

If we have chosen our original four data points correctly, the remaining property to use in interpolation (v) will be bracketed by the values from our first interpolation. This is the case here. Now interpolate using $v = 0.2 \text{ m}^3/\text{kg}$ to get the pressure:

P , kPa	T , °C	v , m ³ /kg
1342	323	0.2

The `h2o.exe` property code gives a pressure of 1336 kPa for this state; the difference is probably due to the error in linear interpolation.

An important point to remember is that when we interpolate, we (implicitly) are holding a variable constant. To get the numbers in the second table we held pressure constant (at the values of 1200 and 1400 kPa). And to get the third table (our sought result) we held temperature constant.

We could have interpolated the first table using $v = 0.2 \text{ m}^3/\text{kg}$ instead of $T = 323^\circ\text{C}$, and then used $T = 323^\circ\text{C}$ to interpolate the second table. The result would be the same in the end.

The general strategy:

As the example illustrates, a double interpolation problem (i.e., a problem in which *neither* of the pair corresponds to a tabulated numerical value) requires that we start with 4 data points. Say, in general, we have a pair of properties x and y , and we want to interpolate to get z . We need to go to the tables and find four data points that are adjacent to x and y . Since the superheated tables are ‘blocked’ according to pressure P , the data points we look up will fall into the following general format:

$P = P_w$			$P = P_e$		
x_{nw}	y_{nw}	z_{nw}	x_{ne}	y_{ne}	z_{ne}
x_{sw}	y_{sw}	z_{sw}	x_{se}	y_{se}	z_{se}

The subscripts in the above refer to ‘north’, ‘south’, ‘east’, and ‘west’. In this sense, the four data points (nw, ne, sw, and se) are points on a map which lie on the vertices of (roughly) a square which encloses the sought data point x, y . We would want, for example,

$$x_{nw} \leq x \leq x_{sw} \quad \text{and} \quad x_{ne} \leq x \leq x_{se}$$

along with

$$y_{nw} \leq y \leq y_{ne} \quad \text{and} \quad y_{sw} \leq y \leq y_{se}$$

although this may not always be the case, as the previous example illustrates.

Once we have located the four data points, the first step is to interpolate holding pressure constant (that is, between the north and south points) with the known value of x . This will give us

$P = P_w$			$P = P_e$		
x	$y_w(x, P_w)$	$z_w(x, P_w)$	x	$y_e(x, P_e)$	$z_e(x, P_e)$

If we have chosen our original points correctly, we would have

$$y_w \leq y \leq y_e$$

We then interpolate using the known value of y to get our sought data point:

$P = P(x, y)$		
x	y	$z(x, y)$

Here is an example:

Find s of water at $v = 0.25 \text{ m}^3/\text{kg}$, $h = 3100 \text{ kJ/kg}$. This again is a superheated state: at $v_g \approx 0.25 \text{ m}^3/\text{kg}$ we see that $h_g \approx 2700 \text{ kJ/kg}$; the water at our actual volume would have a higher energy than at saturation and the state is superheated. Go to the superheated tables, eyeball the numbers for $v = 0.25 \text{ m}^3/\text{kg}$, and find the P region where h is close to 3100 (in this sense, we are using v as the x variable in the general method). At 1000 kPa h will be a little low, and at 1200 it will be a little high. The four data points are:

$P = 1000$			$P = 1200$		
v	h	s	v	h	s
0.2327	2943	6.925	0.2345	3154	7.212
0.2579	3051	7.123	0.2548	3261	7.377

Interpolate holding P constant and using $v = 0.25 \text{ m}^3/\text{kg}$:

$P = 1000$			$P = 1200$		
v	h	s	v	h	s
0.25	3017	7.061	0.25	3236	7.338

Now interpolate using $h = 3100 \text{ kJ/kg}$:

$P = 1076$		
v	h	s
0.25	3100	7.166

The `h2o.exe` code gives $P = 1077 \text{ kPa}$ and $s = 7.173 \text{ kJ/kg K}$ for this state. Again, the error is due to the linear approximation that is inherent in linear interpolation.

Double interpolation in the saturation region

Say we have the following problem:

Find T and x of water for $v = 0.1 \text{ m}^3/\text{kg}$, $u = 1500 \text{ kJ/kg}$: This is a saturated state; at $v_g \approx 0.1 \text{ m}^3/\text{kg}$ the corresponding $u_g \approx 2600 \text{ kJ/kg}$. A saturated vapor at this volume would therefore have a lower energy than the saturated vapor energy, and the system would be a saturated mixture.

Obtaining the state for this pair is more of a rootfinding exercise than an interpolation method. For this particular problem, a strategy would be to

1. Pick a saturation temperature
2. Compute x at this temperature using $v = 0.1 \text{ m}^3/\text{kg}$
3. Use this x to compute u at the temperature.
4. Repeat until we have found two adjacent temperatures which give u 's which bracket 1500 kJ/kg . Interpolate to get the final result.

As you can see, this is a laborious process, although the general procedure is straightforward. The two points in the saturation table which bracket the u value are

T	v	$x(T, v)$	$u(T, v)$
170	0.1	0.409	1479
175	0.1	0.458	1584

Interpolation with $u = 1500$ gives $x = 0.419$, $T = 171^\circ\text{C}$.