Bode Diagram

Frequency Response

- **Frequency Response is the steadystate behavior of the system when forced by a sinusoidal input.**
- **Consider a first order system**

$$G(s) = \frac{C(s)}{R(s)} = \frac{K_p}{\tau s + 1}$$

- **Let us assume that this process is subjected to a sinusoidal input**

$$u(t) = A \sin \omega t$$

where A is the amplitude and ω is the frequency

- **Laplace transform of the input gives**

$$U(s) = \frac{A\omega}{s^2 + \omega^2}$$

- **hence, the Laplace transform of the output becomes**

$$C(s) = \frac{K_p}{\tau s + 1} \frac{A\omega}{s^2 + \omega^2}$$

- **expanded into fractions**

$$C(s) = \frac{B_1}{s + 1/\tau} + \frac{B_2}{s + j\omega} + \frac{B_3}{s - j\omega}$$

- **inverse Laplace transform, take $t \to \infty$,**

$$u(t \to \infty) = \frac{AK_p}{\omega^2\tau^2 + 1}\sin\omega t - \frac{AK_p}{\omega^2\tau^2 + 1}\cos\omega t$$

- **Or**

$$a_1 \sin\omega t + a_2 \cos\omega t = a_3 \sin(\omega t + \varphi)$$

- **where** $a_3 = \sqrt{a_1^2 + a_2^2}, \varphi = \tan^{-1}\left(\frac{a_2}{a_1}\right)$

- **Therefore,** $u(t \to \infty) = \frac{AK_p}{\sqrt{\omega^2\tau^2 + 1}}\sin(\omega t + \varphi)$

$$\varphi = \tan^{-1}(-\omega\tau)$$

- **It can be seen that the response to a sinusoidal input signal is a sinusoidal signal with the same frequency but a different angle. The output signal lags behind the input signal by an angle φ, which depends on the frequency ω.**
- **The ratio between the amplitude of the input sine wave and the output sine wave, called amplitude ratio,**

$$AR = \frac{K_p}{\sqrt{\omega^2\tau^2 + 1}}$$

- **Substitution of $s = j\omega$ in the transfer function $G(s)$ results in a transfer function in the frequency domain $G(j\omega)$. Then the magnitude is equal to the amplitude ratio AR:**

$$AR = \left|G(j\omega)\right|$$

- **and the phase angle or argument is equal to:**

$$\varphi = \arg\left[G(j\omega)\right]$$

- ***G*(*jω*) is a complex number, it can therefore be represented by a real and imaginary part:**

$$G(j\omega) = \mathrm{Re}[G(j\omega)] + j\,\mathrm{Im}[G(j\omega)]$$

- **for which:**

$$AR = |G(j\omega)| = \sqrt{(\mathrm{Re}[G(j\omega)])^2 + (j\,\mathrm{Im}[G(j\omega)])^2}$$

$$\varphi = \arg[G(j\omega)] = \tan^{-1}\frac{\mathrm{Im}[G(j\omega)]}{\mathrm{Re}[G(j\omega)]}$$

- **When substituting $s = j\omega$ into**

$$G(j\omega) = \frac{C(j\omega)}{R(j\omega)} = \frac{K_p}{\tau(j\omega)+1}\frac{\left(-\tau(j\omega)+1\right)}{\left(-\tau(j\omega)+1\right)}$$

$$= \frac{K_p}{\omega^2\tau^2+1} - j\frac{K_p\omega\tau}{\omega^2\tau^2+1}$$

- **Therefore,** $magnitude = AR = \dfrac{K_p}{\sqrt{\omega^2\tau^2+1}}$

$$phase\ angle\ \ \varphi = \tan^{-1}\left(-\omega\tau\right)$$

- **A transfer function is often a combination of sub-transfer functions, consisting of numerator and denominator terms:**

$$G(j\omega)=\frac{G_1(j\omega)G_2(j\omega)\cdots G_n(j\omega)}{G_{n+1}(j\omega)G_{n+2}(j\omega)\cdots G_m(j\omega)}$$

- **It can easily be shown that**

$$AR=\left|G_1(j\omega)G_2(j\omega)\cdots\right|=\left|G_1(j\omega)\right|\left|G_2(j\omega)\right|\cdots$$

$$\varphi=\arg\left[G_1(j\omega)G_2(j\omega)\cdots\right]=\arg\left[G_1(j\omega)\right]+\arg\left[G_2(j\omega)\right]+\cdots$$

- **For inverse transfer functions the amplitude ratio and phase angle can be derived from the property:**

$$AR = \left|\frac{1}{G(j\omega)}\right| = \frac{1}{|G(j\omega)|} \qquad \varphi = \arg\frac{1}{G(j\omega)} = -\arg[G(j\omega)]$$

- **Therefore,**

$$AR = |G(j\omega)| = \frac{\prod_{i=1}^{n} |G_i(j\omega)|}{\prod_{j=n+1}^{m} |G_j(j\omega)|}$$

$$\varphi = \arg[G(j\omega)] = \sum_{i}^{n} \arg[G_i(j\omega)] - \sum_{j=n+1}^{m} \arg[G_j(j\omega)]$$

Bode Diagram

- **The graphs in which the amplitude ratio and phase shift are plotted as a function of the frequency ω, are called Bode diagrams.**
- **In the Bode plot, log(AR) and φ are shown as a function of ω. In case of the first-order process,**

$$magnitude = AR = \frac{K_p}{\sqrt{\omega^2\tau^2+1}}$$

- **Or** $$\log(AR) = \log(K_p) - \frac{1}{2}\log(\omega^2\tau^2+1)$$

- **thus log (*AR*) becomes a linear function of log($\omega\tau$) with a slope of –1.**
- **In case of $\omega << 1/\tau$:**

$$\log(AR) = \log(K_p) + \frac{1}{2}\log(1) = \log(K_p)$$

thus the gain of the process is independent of the frequency ω .

- **In case $\omega = 1/\tau$:**

$$\log(AR) = \log(K_p) + \frac{1}{2}\log(2)$$

$$phase\ angle\ \ \varphi = \tan^{-1}\left(-\omega\tau\right)$$

- **In case of $\omega >> 1/\tau$:**

$$phase\ angle\ \ \varphi = \tan^{-1}\left(\infty\right) = -90^{o}$$

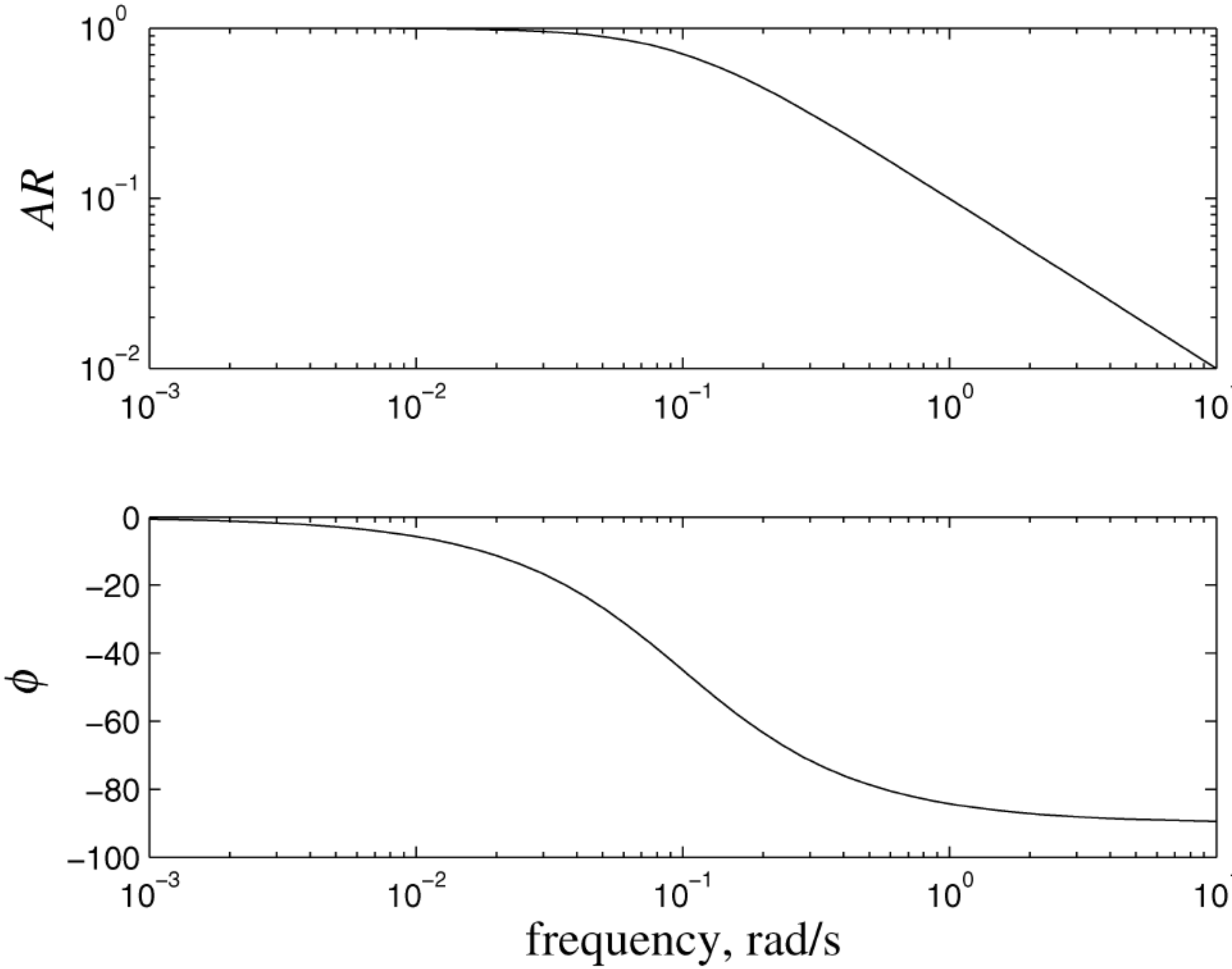
- **In case $\omega << 1/\tau$:**

$$phase\ angle\ \ \varphi = \tan^{-1}\left(0\right) = 0^{o}$$

- **In case $\omega = 1/\tau$:**

$$phase\ angle\ \ \varphi = \tan^{-1}\left(1\right) = -45^{o}$$

Bode diagram of a first-order process



Second-order Non-interacting System

$$G(s)=\frac{K_1}{(\tau_1 s+1)}\frac{K_2}{(\tau_2 s+1)}$$

- **the amplitude ratio and phase angle become:**

$$AR=|G(j\omega)|=\frac{\prod_{i=1}^{n}|G_i(j\omega)|}{\prod_{j=n+1}^{m}|G_j(j\omega)|}$$

$$AR=\frac{K_1K_2}{\sqrt{\omega^2\tau_1^2+1}\sqrt{\omega^2\tau_2^2+1}}$$

$$\varphi=\arg[G(j\omega)]=\sum_{i}^{n}\arg[G_i(j\omega)]-\sum_{j=n+1}^{m}\arg[G_j(j\omega)]$$

$$\varphi=\tan^{-1}(-\omega\tau_1)+\tan^{-1}(-\omega\tau_2)$$

Underdamped Second-order System

- **Second-order System**

$$G(s) = \frac{K_p}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

- **Substituting $s = j\omega$ into equation and rearranging results in:**

$$G(j\omega) = \frac{K_p}{(1 - \omega^2\tau^2) + 2j\zeta\omega\tau}$$

- **Recall:**

$$AR = |G(j\omega)| = \sqrt{(\mathrm{Re}[G(j\omega)])^2 + (j\,\mathrm{Im}[G(j\omega)])^2}$$

$$\varphi = \arg[G(j\omega)] = \tan^{-1}\frac{\mathrm{Im}[G(j\omega)]}{\mathrm{Re}[G(j\omega)]}$$

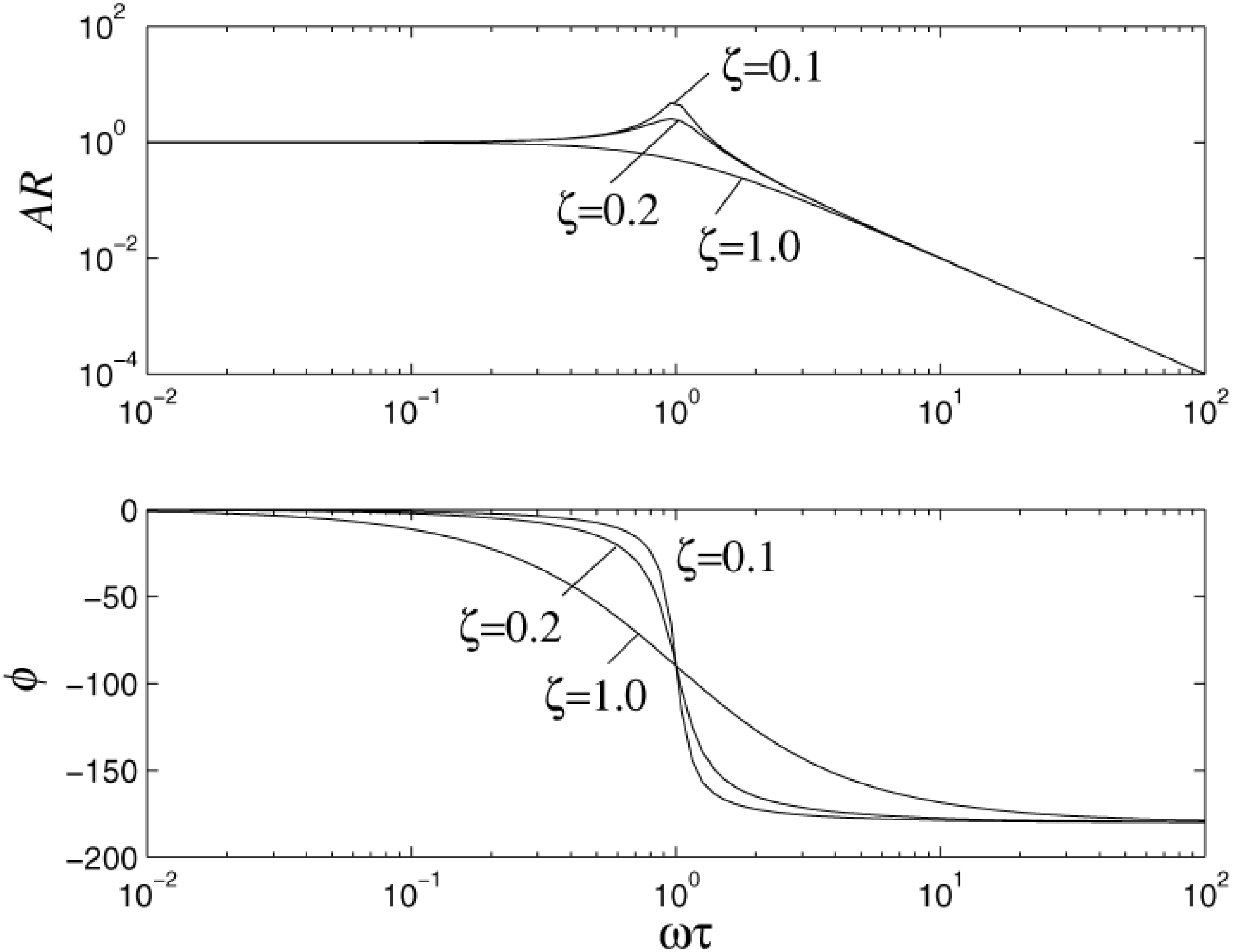
- **Therefore,** $$G(j\omega) = \frac{K_p}{\left(1-\omega^2\tau^2\right)+2j\zeta\omega\tau}\frac{\left(1-\omega^2\tau^2\right)-2j\zeta\omega\tau}{\left(1-\omega^2\tau^2\right)-2j\zeta\omega\tau}$$

$$= \frac{K_p}{\left(1-\omega^2\tau^2\right)^2+\left(2\zeta\omega\tau\right)^2}\left[\left(1-\omega^2\tau^2\right)-2j\zeta\omega\tau\right]$$

$$AR = \frac{K_p}{\sqrt{\left(1-\omega^2\tau^2\right)^2+\left(2\zeta\omega\tau\right)^2}}$$

$$\varphi = \tan^{-1}\left(-\frac{2\zeta\omega\tau}{1-\omega^2\tau^2}\right)$$

Bode diagram for second-order under-damped process



- **On the Bode magnitude plot, decibels are used, defined as:**

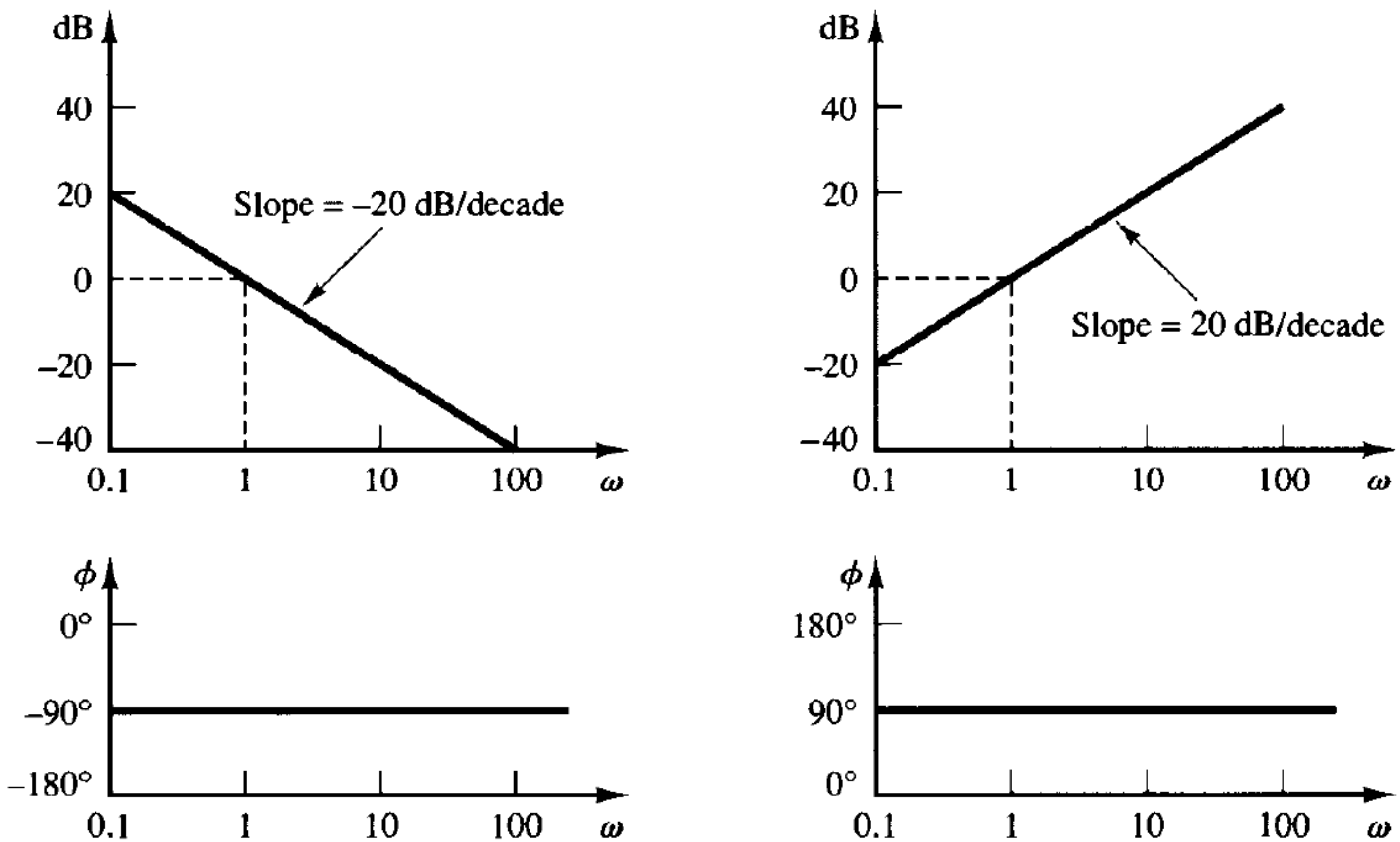
$$a_{dB} = 20\log_{10} a, \qquad decibels\ or\ dB$$

- **The log-magnitude curve for a constant gain *K* is a horizontal straight line at the magnitude of**

$$20\log_{10} K,\ \ dB$$

- **Integral and derivative factors** $(j\omega)^{\mp}$

$$20\log\left|(j\omega)^{\mp}\right| = \mp 20\log\omega,\ \ dB$$



Bode diagram of
$G(j\omega) = 1/j\omega$

(a)

Bode diagram of
$G(j\omega) = j\omega$

(b)

Example I

- **For the transfer function:**

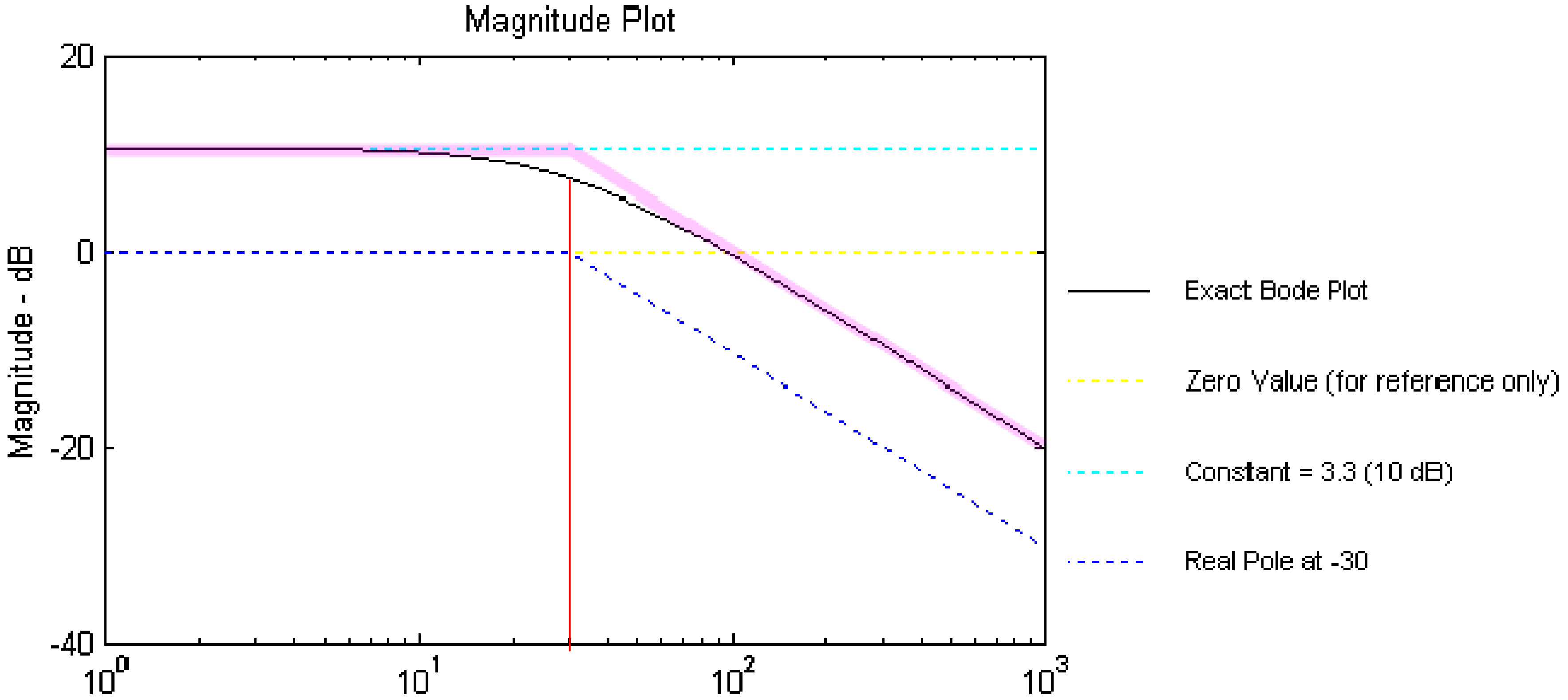
$$H(s) = \frac{100}{s + 30}$$

- **Step 1: Rewrite the transfer function in proper form. Make both the lowest order term in the numerator and denominator unity. Therefore, the numerator is an order 0 polynomial, the denominator is order 1.**

$$H(s) = \frac{100}{30}\frac{1}{\frac{s}{30}+1} = 3.3\frac{1}{\frac{s}{30}+1}$$

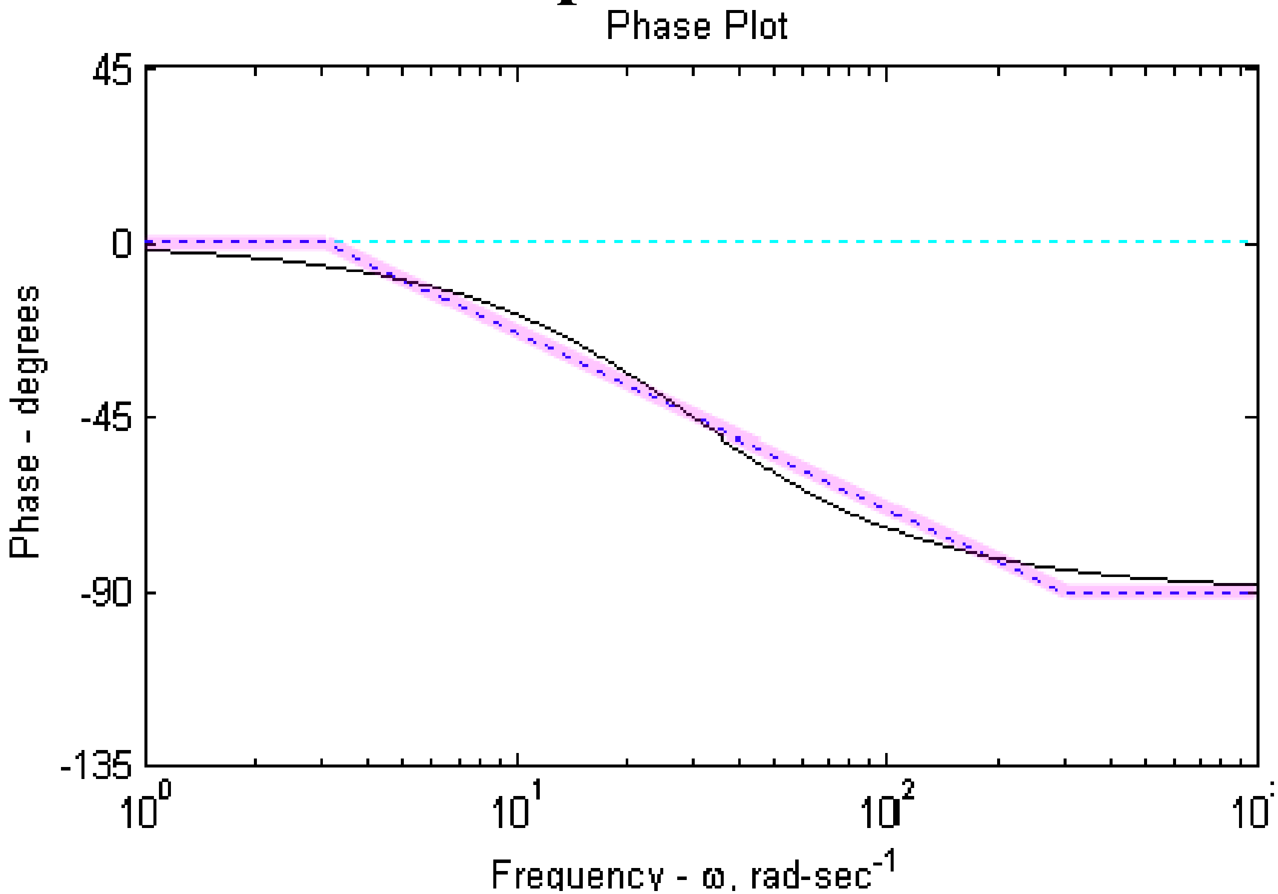
- **Step 2: Separate the transfer function into its constituent parts.**
- **The transfer function has 2 components:**
  - **A constant of 3.3**
  - **A pole at s=-30**

- **Step 3: Draw the Bode diagram for each part.**



- **The constant is the cyan (青色) line (A quantity of 3.3 is equal to 10.4 dB). The phase is constant at 0 degrees.**
- **The pole at -30 rad/sec is the blue line. It is 0 dB up to the break frequency, then drops off with a slope of -20 dB/dec.**
- **The phase is 0 degrees up to 1/10 the break frequency (3 rad/sec) then drops linearly down to -90 degrees at 10 times the break frequency (300 rad/sec).**

Step 4: Draw the overall Bode diagram by adding up the results from step 3.



Example II

- **Bode Diagram for the transfer function:**

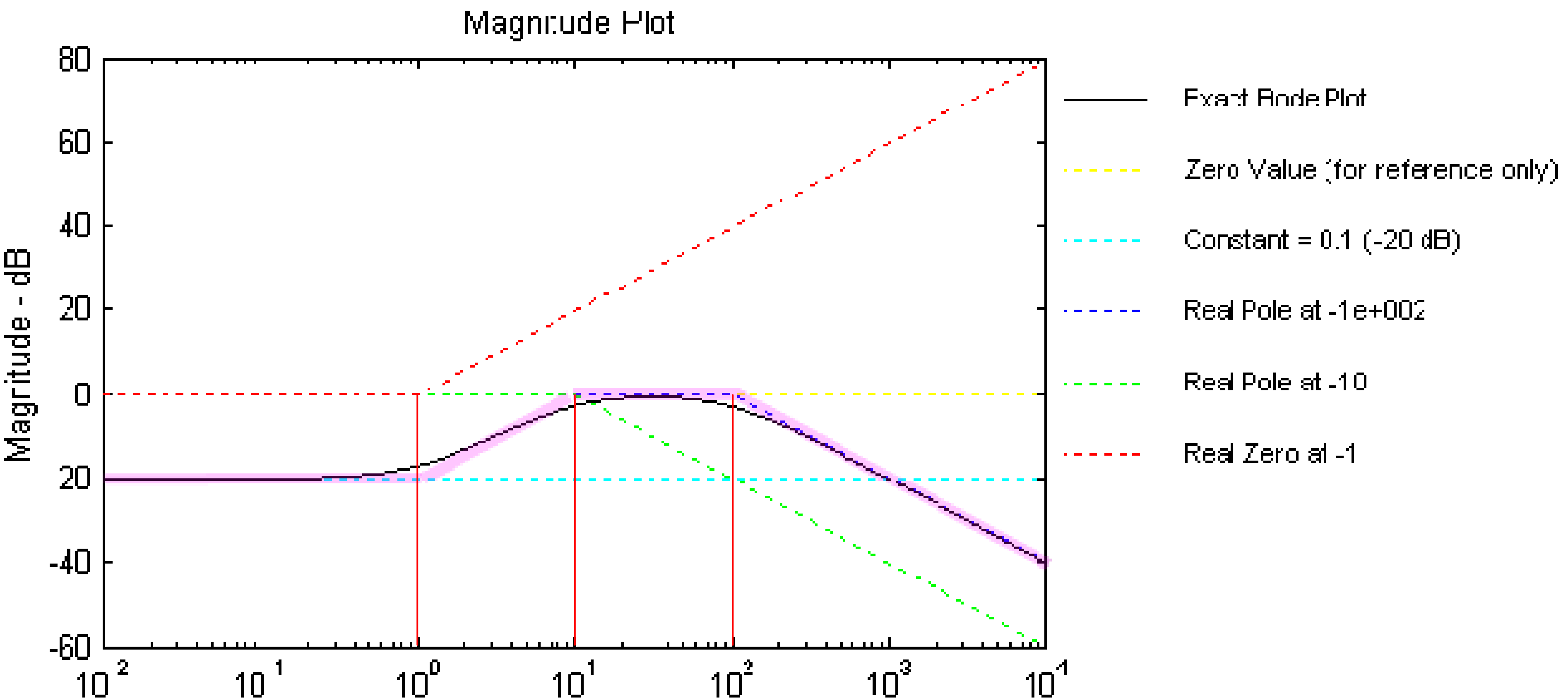
$$H(s) = 100\frac{s+1}{(s+10)(s+100)} = 100\frac{s+1}{s^2+110s+1000}$$

- **Step 1: Rewrite the transfer function in proper form. Make both the lowest order term in the numerator and denominator unity. The numerator is an order 1 polynomial, the denominator is order 2.**

$$H(s) = \frac{100}{10\cdot 100}\frac{\frac{s}{1}+1}{\left(\frac{s}{10}+1\right)\left(\frac{s}{100}+1\right)} = 0.1\frac{\frac{s}{1}+1}{\left(\frac{s}{10}+1\right)\left(\frac{s}{100}+1\right)}$$

- **Step 2: Separate the transfer function into its constituent parts.**
- **The transfer function has 4 components:**
  - **A constant of 0.1**
  - **A pole at s=-10**
  - **A pole at s=-100**
  - **A zero at s=-1**

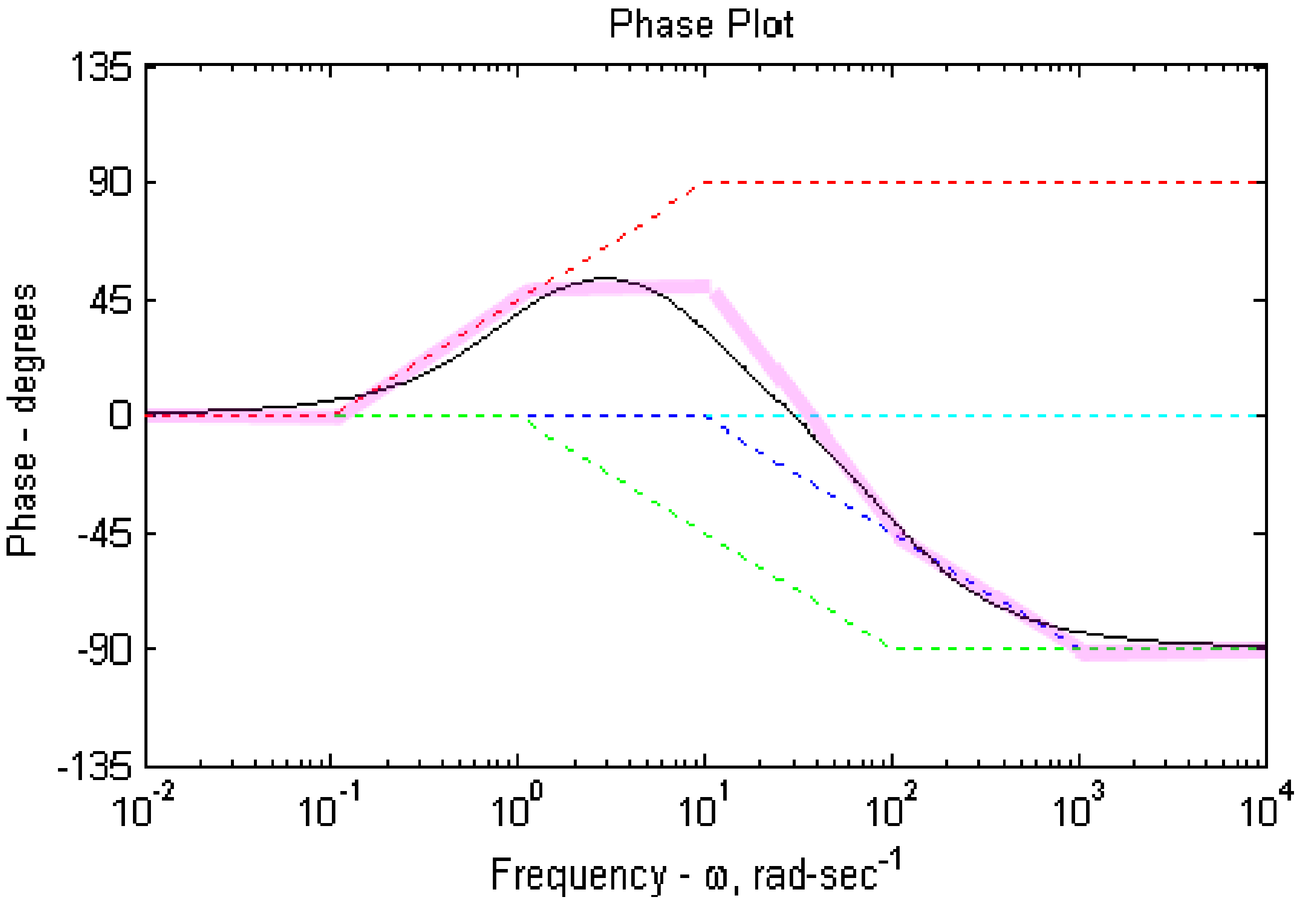
- **Step 3: Draw the Bode diagram for each part.**
- **The constant is the cyan line (A quantity of 0.1 is equal to -20 dB). The phase is constant at 0 degrees.**
- **The zero at 1 rad/sec is the red line. It is 0 dB up to the break frequency, then rises at 20 dB/dec. The phase is 0 degrees up to 1/10 the break frequency (0.1 rad/sec) then rises linearly to 90 degrees at 10 times the break frequency (10 rad/sec).**



Magnitude Plot
Magnitude - dB
80
60
40
20
0
20
-40
-60
10 2
10 1
10 0
10 1
10 2
10 3
10 4
Exact Bode Plot
Zero Value (for reference only)
Constant = 0.1 (-20 dB)
Real Pole at -1e+002
Real Pole at -10
Real Zero at -1

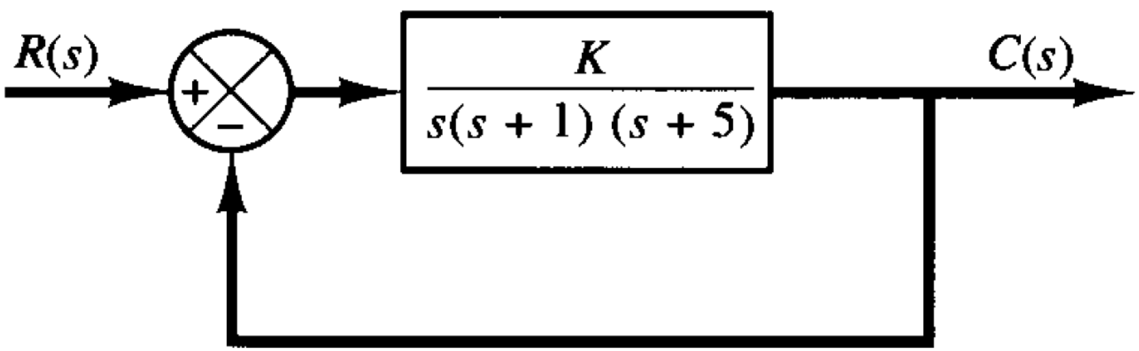
- **The pole at 10 rad/sec is the green line. It is 0 dB up to the break frequency, then drops off with a slope of -20 dB/dec. The phase is 0 degrees up to 1/10 the break frequency (1 rad/sec) then drops linearly down to -90 degrees at 10 times the break frequency (100 rad/sec).**
- **The pole at 100 rad/sec is the blue line. It is 0 dB up to the break frequency, then drops off with a slope of -20 dB/dec. The phase is 0 degrees up to 1/10 the break frequency (10 rad/sec) then drops linearly down to -90 degrees at 10 times the break frequency (1000 rad/sec).**

Step 4: Draw the overall Bode diagram by adding up the results from step 3.

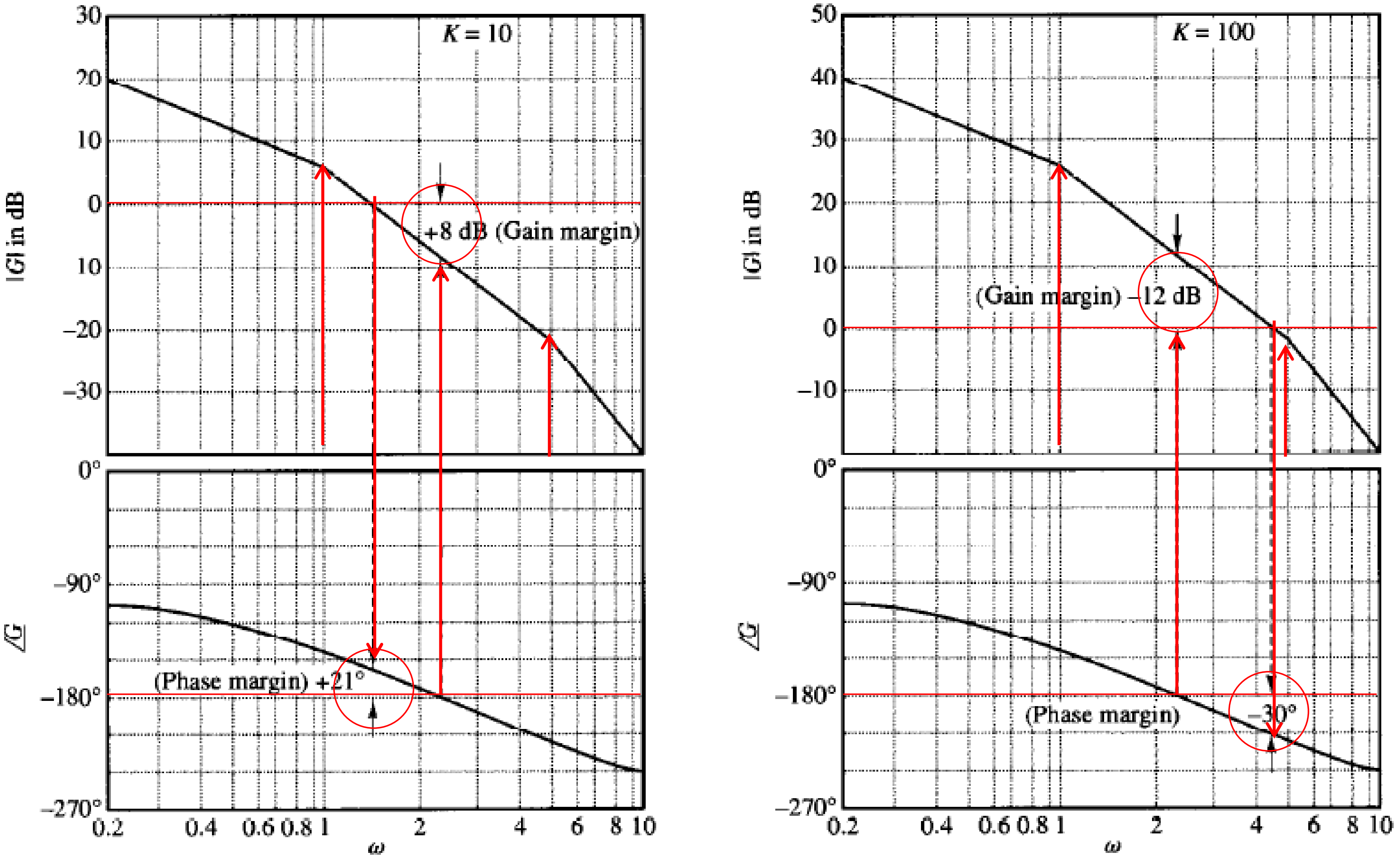


Stability Analysis

- The phase and gain margins can easily be obtained from the Bode diagram.



- For K = 10 and K = 100, find the phase and gain margins



K = 10
|G| in dB
30
20
10
0
–10
–20
–30
+8 dB (Gain margin)
∠G
0°
–90°
–180°
–270°
(Phase margin) +21°
0.2
0.4
0.6 0.8 1
2
4
6
8 10
ω
K = 100
|G| in dB
50
40
30
20
10
0
–10
(Gain margin) –12 dB
∠G
0°
–90°
–180°
–270°
(Phase margin)
–30°
0.2
0.4
0.6 0.8 1
2
4
6
8 10
ω