

Bode Diagram

Min Huang, PhD
CheEng@TongjiU

Frequency Response

- Frequency Response is the **steadystate** behavior of the system when forced by a sinusoidal input.
- Consider a first order system

$$G(s) = \frac{C(s)}{R(s)} = \frac{K_p}{\tau s + 1}$$

- Let us assume that this process is subjected to a sinusoidal input

$$u(t) = A \sin \omega t$$

where A is the amplitude and ω is the frequency

- **Laplace transform of the input gives**

$$U(s) = \frac{A\omega}{s^2 + \omega^2}$$

- **hence, the Laplace transform of the output becomes**

$$C(s) = \frac{K_p}{\tau s + 1} \frac{A\omega}{s^2 + \omega^2}$$

- **expanded into fractions**

$$C(s) = \frac{B_1}{s + 1/\tau} + \frac{B_2}{s + j\omega} + \frac{B_3}{s - j\omega}$$

- **inverse Laplace transform, take $t \rightarrow \infty$,** **steadystate**

$$u(t \rightarrow \infty) = \frac{AK_p}{\omega^2 \tau^2 + 1} \sin \omega t - \frac{AK_p}{\omega^2 \tau^2 + 1} \cos \omega t$$

- **Or** $a_1 \sin \omega t + a_2 \cos \omega t = a_3 \sin(\omega t + \varphi)$

- **where** $a_3 = \sqrt{a_1^2 + a_2^2}$, $\varphi = \tan^{-1}\left(\frac{a_2}{a_1}\right)$

- **Therefore,** $u(t \rightarrow \infty) = \frac{AK_p}{\sqrt{\omega^2 \tau^2 + 1}} \sin(\omega t + \varphi)$

$$\varphi = \tan^{-1}(-\omega\tau)$$

- It can be seen that the response to a sinusoidal input signal is a sinusoidal signal with the **same frequency but a different angle**. The output signal **lags behind** the input signal **by an angle φ** , which **depends on** the frequency ω .
- The ratio between the amplitude of the input sine wave and the output sine wave, called **amplitude ratio**,

$$AR = \frac{K_p}{\sqrt{\omega^2 \tau^2 + 1}}$$

- **Substitution of $s = j\omega$ in the transfer function $G(s)$ results in a transfer function in the frequency domain $G(j\omega)$. Then the magnitude is equal to the amplitude ratio AR :**

$$AR = |G(j\omega)|$$

- **and the phase angle or argument is equal to:**

$$\varphi = \arg[G(j\omega)]$$

- **$G(j\omega)$ is a complex number, it can therefore be represented by a real and imaginary part:**

$$G(j\omega) = \text{Re}[G(j\omega)] + j \text{Im}[G(j\omega)]$$

- **for which:**

$$AR = |G(j\omega)| = \sqrt{(\text{Re}[G(j\omega)])^2 + (j \text{Im}[G(j\omega)])^2}$$

$$\varphi = \arg[G(j\omega)] = \tan^{-1} \frac{\text{Im}[G(j\omega)]}{\text{Re}[G(j\omega)]}$$

- **When substituting $s = j\omega$ into**

$$G(j\omega) = \frac{C(j\omega)}{R(j\omega)} = \frac{K_p}{\tau(j\omega)+1} \frac{(-\tau(j\omega)+1)}{(-\tau(j\omega)+1)}$$

$$= \frac{K_p}{\omega^2 \tau^2 + 1} - j \frac{K_p \omega \tau}{\omega^2 \tau^2 + 1}$$

- **Therefore,** *magnitude* $= AR = \frac{K_p}{\sqrt{\omega^2 \tau^2 + 1}}$

$$\textit{phase angle } \varphi = \tan^{-1}(-\omega\tau)$$

- **A transfer function is often a combination of sub-transfer functions, consisting of numerator and denominator terms:**

$$G(j\omega) = \frac{G_1(j\omega)G_2(j\omega)\cdots G_n(j\omega)}{G_{n+1}(j\omega)G_{n+2}(j\omega)\cdots G_m(j\omega)}$$

- **It can easily be shown that**

$$AR = |G_1(j\omega)G_2(j\omega)\cdots| = |G_1(j\omega)||G_2(j\omega)|\cdots$$

$$\varphi = \arg[G_1(j\omega)G_2(j\omega)\cdots] = \arg[G_1(j\omega)] + \arg[G_2(j\omega)] + \cdots$$

- **For inverse transfer functions the amplitude ratio and phase angle can be derived from the property:**

$$AR = \left| \frac{1}{G(j\omega)} \right| = \frac{1}{|G(j\omega)|} \quad \varphi = \arg \frac{1}{G(j\omega)} = -\arg[G(j\omega)]$$

- **Therefore,**
$$AR = |G(j\omega)| = \frac{\prod_{i=1}^n |G_i(j\omega)|}{\prod_{j=n+1}^m |G_j(j\omega)|}$$

$$\varphi = \arg[G(j\omega)] = \sum_i^n \arg[G_i(j\omega)] - \sum_{j=n+1}^m \arg[G_j(j\omega)]$$

Bode Diagram

- The graphs in which the amplitude ratio and phase shift are plotted as a function of the frequency ω , are called Bode diagrams.
- In the Bode plot, $\log(AR)$ and φ are shown as a function of ω . In case of the first-order process,

$$\text{magnitude} = AR = \frac{K_p}{\sqrt{\omega^2 \tau^2 + 1}}$$

- Or $\log(AR) = \log(K_p) - \frac{1}{2} \log(\omega^2 \tau^2 + 1)$

- thus $\log (AR)$ becomes a linear function of $\log(\omega\tau)$ with a slope of -1 .

- In case of $\omega \ll 1/\tau$:

$$\log(AR) = \log(K_p) + \frac{1}{2} \log(1) = \log(K_p) \quad \text{Slope} = 0$$

thus the gain of the process is independent of the frequency ω .

- In case $\omega = 1/\tau$:

$$\log(AR) = \log(K_p) + \frac{1}{2} \log(2)$$

- In case of $\omega \gg 1/\tau$:

Slope = -1

$$\log(AR) = \log(K_p) - \frac{1}{2} \log(\omega^2 \tau^2 + 1) \approx \log(K_p) - \log \omega - \log \tau$$

$$\text{phase angle } \varphi = \tan^{-1}(-\omega\tau)$$

- **In case of $\omega \gg 1/\tau$:**

$$\text{phase angle } \varphi = \tan^{-1}(\infty) = -90^\circ$$

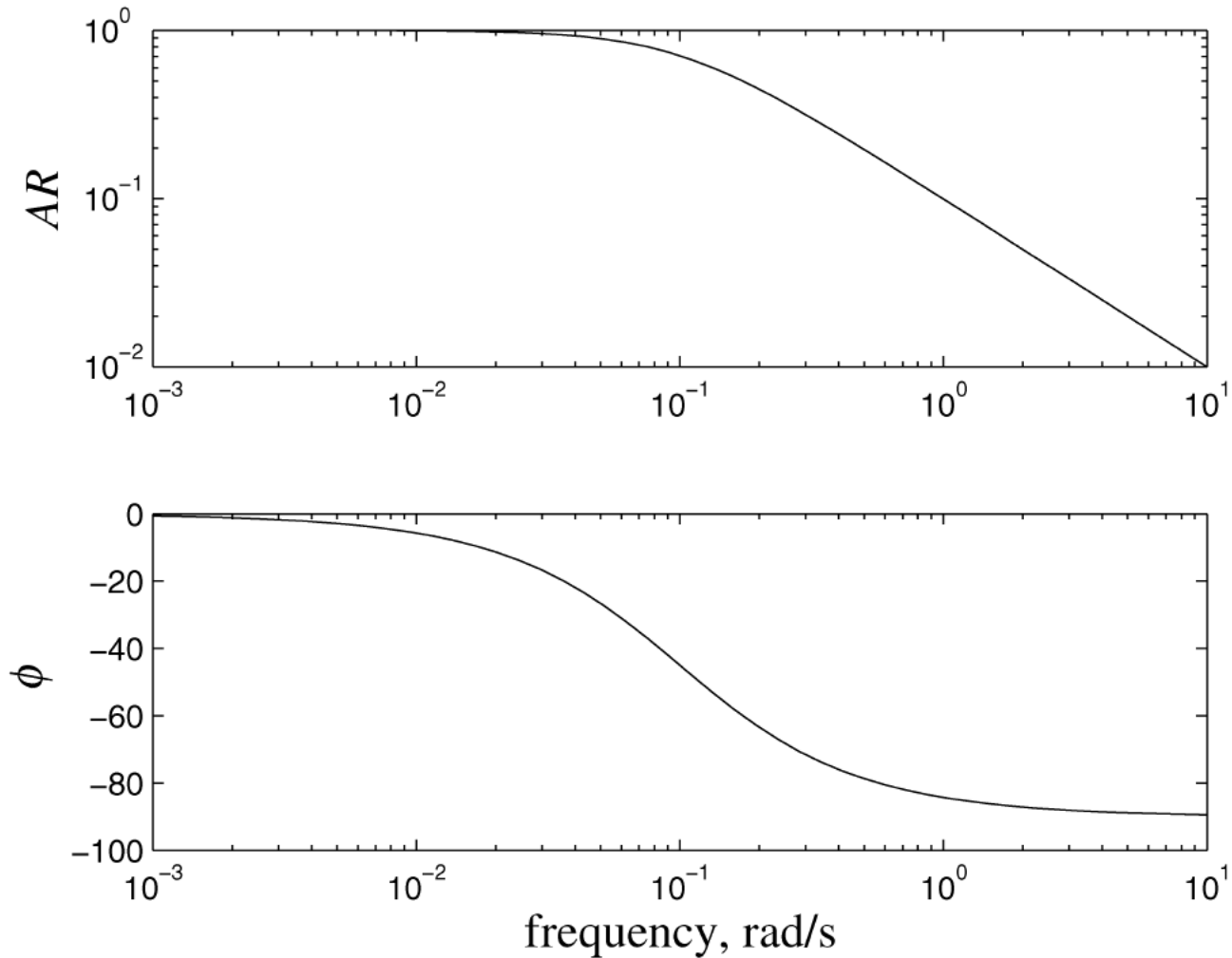
- **In case $\omega \ll 1/\tau$:**

$$\text{phase angle } \varphi = \tan^{-1}(0) = 0^\circ$$

- **In case $\omega = 1/\tau$:**

$$\text{phase angle } \varphi = \tan^{-1}(1) = -45^\circ$$

Bode diagram of a first-order process



Second-order Non-interacting System

$$G(s) = \frac{K_1}{(\tau_1 s + 1)} \frac{K_2}{(\tau_2 s + 1)}$$

- the amplitude ratio and phase angle become:

$$AR = |G(j\omega)| = \frac{\prod_{i=1}^n |G_i(j\omega)|}{\prod_{j=n+1}^m |G_j(j\omega)|}$$

$$AR = \frac{K_1 K_2}{\sqrt{\omega^2 \tau_1^2 + 1} \sqrt{\omega^2 \tau_2^2 + 1}}$$

$$\varphi = \arg[G(j\omega)] = \sum_i \arg[G_i(j\omega)] - \sum_{j=n+1}^m \arg[G_j(j\omega)]$$

$$\varphi = \tan^{-1}(-\omega\tau_1) + \tan^{-1}(-\omega\tau_2)$$

Underdamped Second-order System

- **Second-order System**

$$G(s) = \frac{K_p}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

- **Substituting $s = j\omega$ into equation and rearranging results in:**

$$G(j\omega) = \frac{K_p}{(1 - \omega^2\tau^2) + 2j\zeta\omega\tau}$$

- **Recall:**

$$AR = |G(j\omega)| = \sqrt{(\text{Re}[G(j\omega)])^2 + (j \text{Im}[G(j\omega)])^2}$$

$$\varphi = \arg[G(j\omega)] = \tan^{-1} \frac{\text{Im}[G(j\omega)]}{\text{Re}[G(j\omega)]}$$

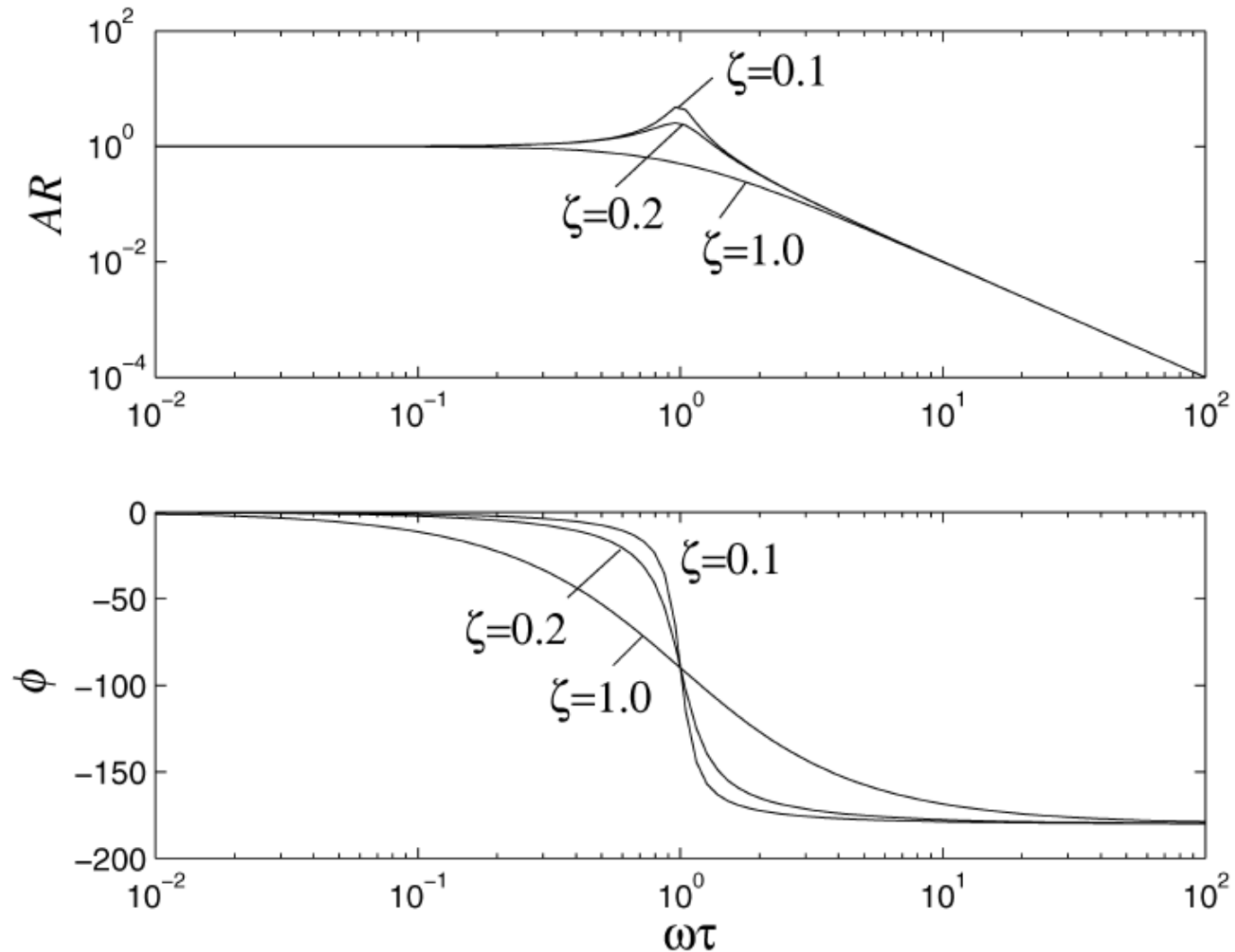
• **Therefore,**
$$G(j\omega) = \frac{K_p}{(1 - \omega^2 \tau^2) + 2j\zeta\omega\tau} \frac{(1 - \omega^2 \tau^2) - 2j\zeta\omega\tau}{(1 - \omega^2 \tau^2) - 2j\zeta\omega\tau}$$

$$= \frac{K_p}{(1 - \omega^2 \tau^2)^2 + (2\zeta\omega\tau)^2} [(1 - \omega^2 \tau^2) - 2j\zeta\omega\tau]$$

$$AR = \frac{K_p}{\sqrt{(1 - \omega^2 \tau^2)^2 + (2\zeta\omega\tau)^2}}$$

$$\varphi = \tan^{-1} \left(-\frac{2\zeta\omega\tau}{1 - \omega^2 \tau^2} \right)$$

Bode diagram for second-order under-damped process



- On the Bode magnitude plot, decibels are used, **defined** as:

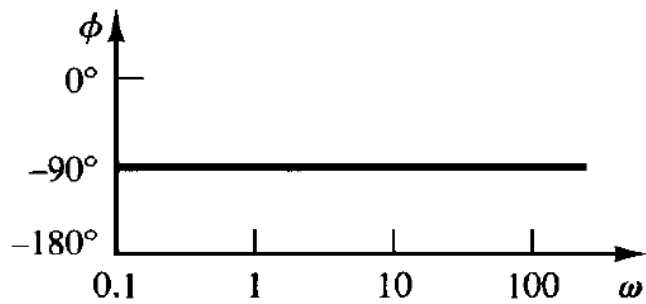
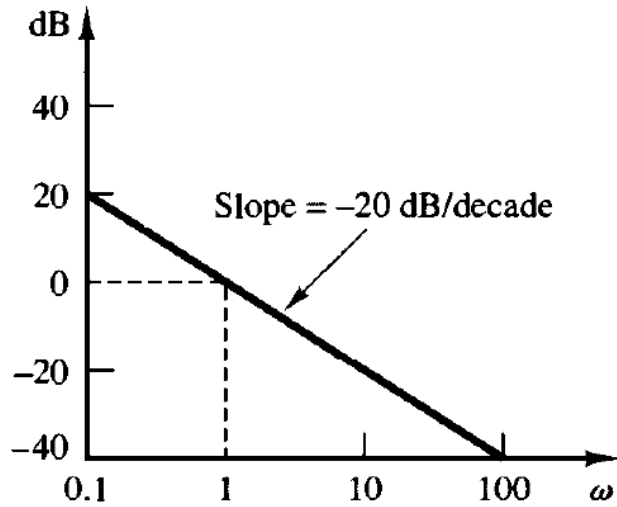
$$a_{dB} = 20 \log_{10} a, \quad \text{decibels or dB}$$

- The log-magnitude curve for a constant gain K is a **horizontal straight line** at the magnitude of

$$20 \log_{10} K, \quad \text{dB}$$

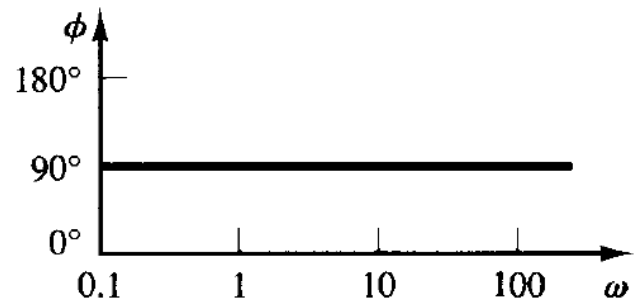
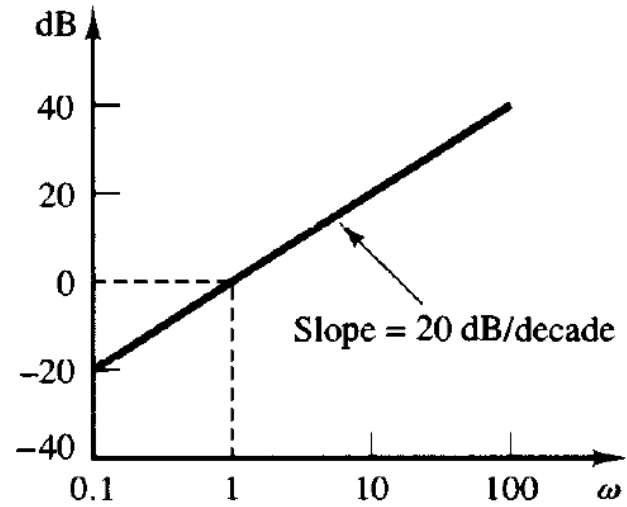
- Integral and derivative factors $(j\omega)^{\mp}$

$$20 \log |(j\omega)^{\mp}| = \mp 20 \log \omega, \quad \text{dB}$$



Bode diagram of
 $G(j\omega) = 1/j\omega$

(a)



Bode diagram of
 $G(j\omega) = j\omega$

(b)

Example I

- For the transfer function:

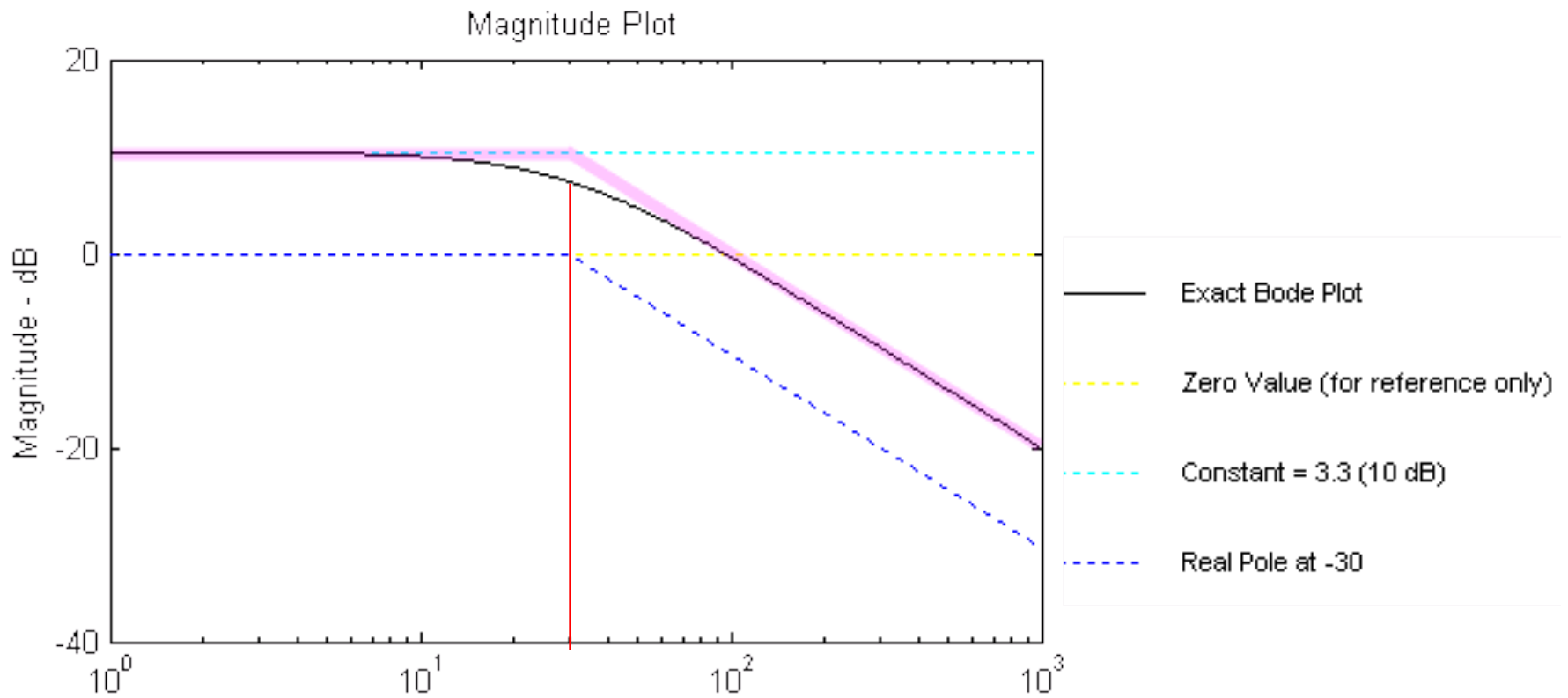
$$H(s) = \frac{100}{s + 30}$$

- **Step 1: Rewrite the transfer function in **proper form**. Make both the lowest order term in the numerator and denominator unity. Therefore, the numerator is an order 0 polynomial, the denominator is order 1.**

$$H(s) = \frac{100}{30} \frac{1}{\frac{s}{30} + 1} = 3.3 \frac{1}{\frac{s}{30} + 1}$$

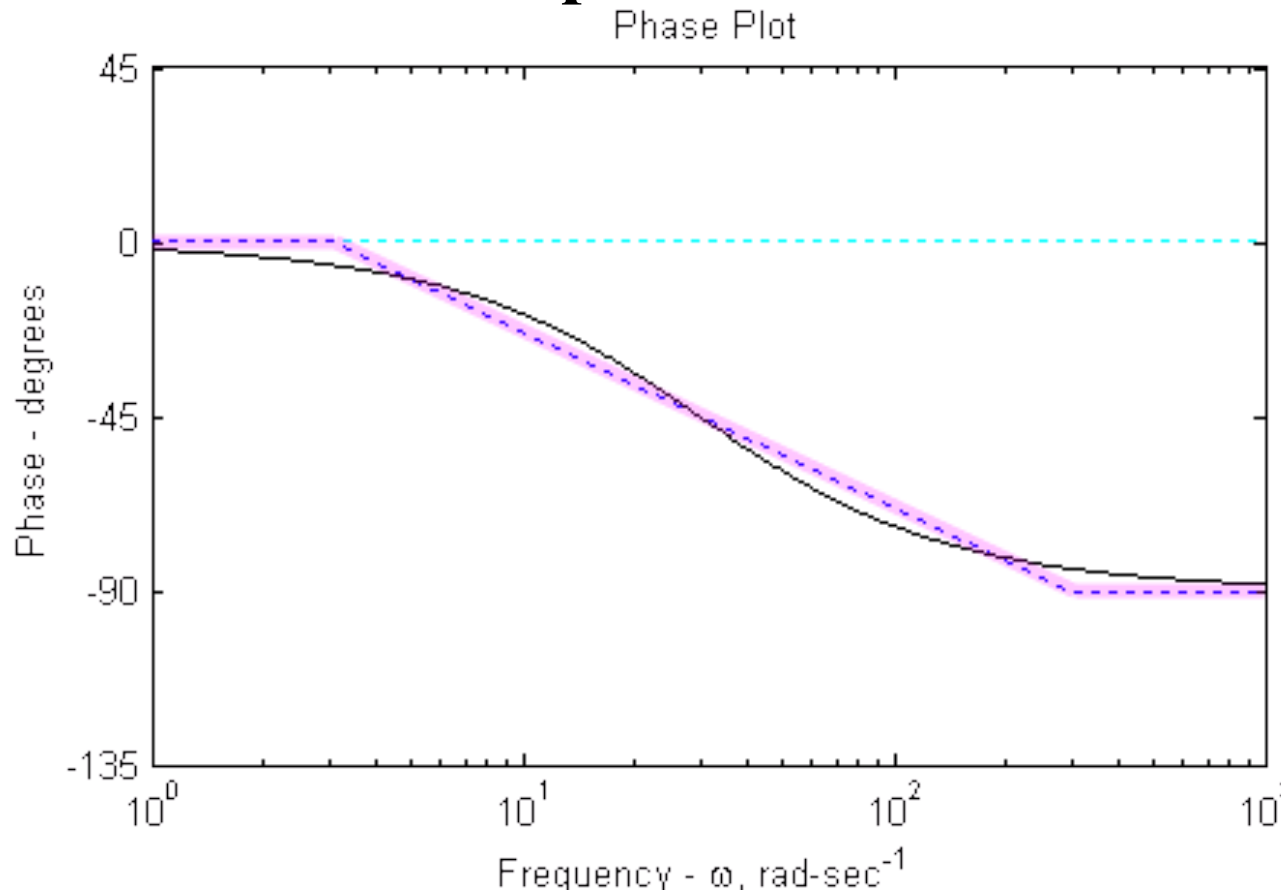
- **Step 2: Separate the transfer function into its constituent parts.**
- **The transfer function has 2 components:**
 - **A constant of 3.3**
 - **A pole at $s=-30$**

- **Step 3: Draw the Bode diagram for each part.**



- The constant is the **cyan (青色) line** (A quantity of 3.3 is equal to 10.4 dB). The phase is constant at 0 degrees.
- The pole at -30 rad/sec is the **blue line**. It is 0 dB up to the **break frequency**, then drops off with a **slope of -20 dB/dec**.
- The phase is 0 degrees up to **1/10** the break frequency (3 rad/sec) then drops linearly down to -90 degrees at **10 times** the break frequency (300 rad/sec).

Step 4: Draw the overall Bode diagram by adding up the results from step 3.



Example II

- **Bode Diagram for the transfer function:**

$$H(s) = 100 \frac{s+1}{(s+10)(s+100)} = 100 \frac{s+1}{s^2 + 110s + 1000}$$

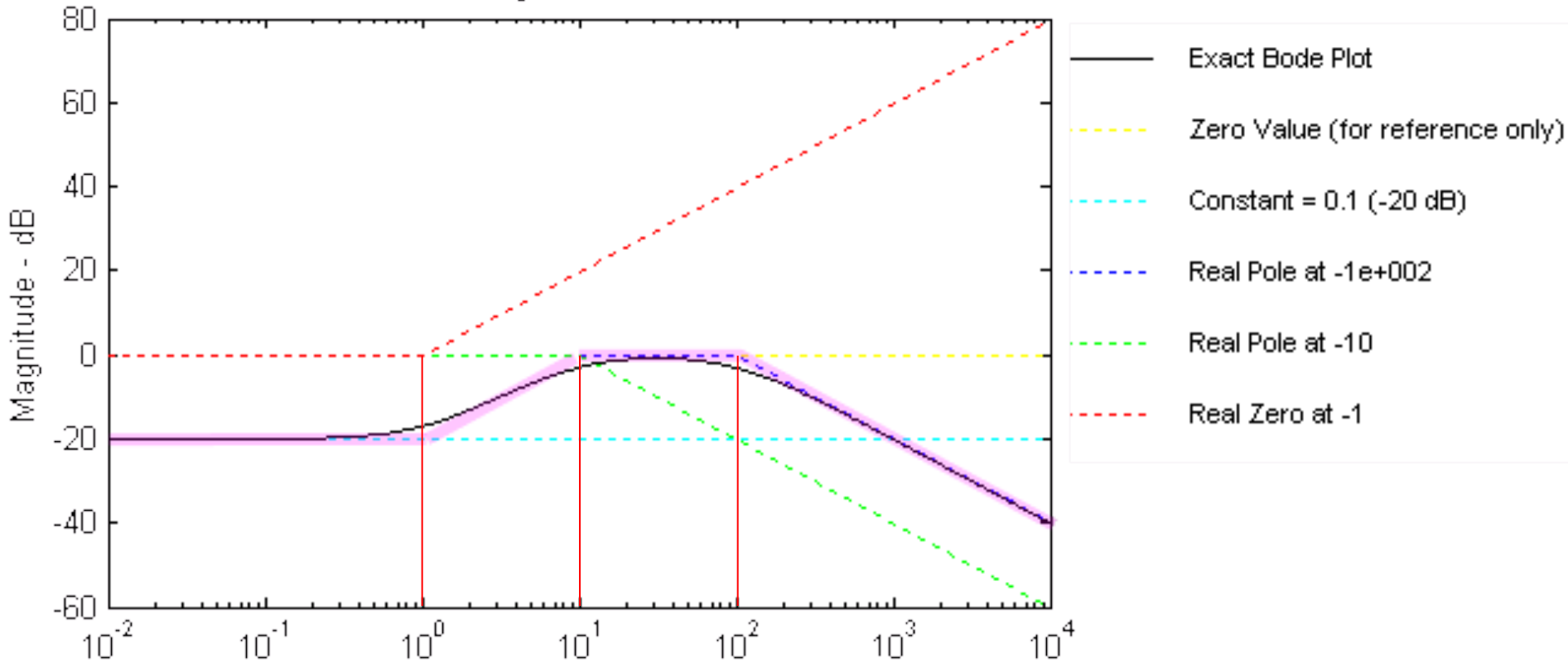
- **Step 1: Rewrite the transfer function in proper form. Make both the lowest order term in the numerator and denominator unity. The numerator is an order 1 polynomial, the denominator is order 2.**

$$H(s) = \frac{100}{10 \cdot 100} \frac{\frac{s}{1} + 1}{\left(\frac{s}{10} + 1\right)\left(\frac{s}{100} + 1\right)} = 0.1 \frac{\frac{s}{1} + 1}{\left(\frac{s}{10} + 1\right)\left(\frac{s}{100} + 1\right)}$$

- **Step 2: Separate the transfer function into its constituent parts.**
- **The transfer function has 4 components:**
 - **A constant of 0.1**
 - **A pole at $s=-10$**
 - **A pole at $s=-100$**
 - **A zero at $s=-1$**

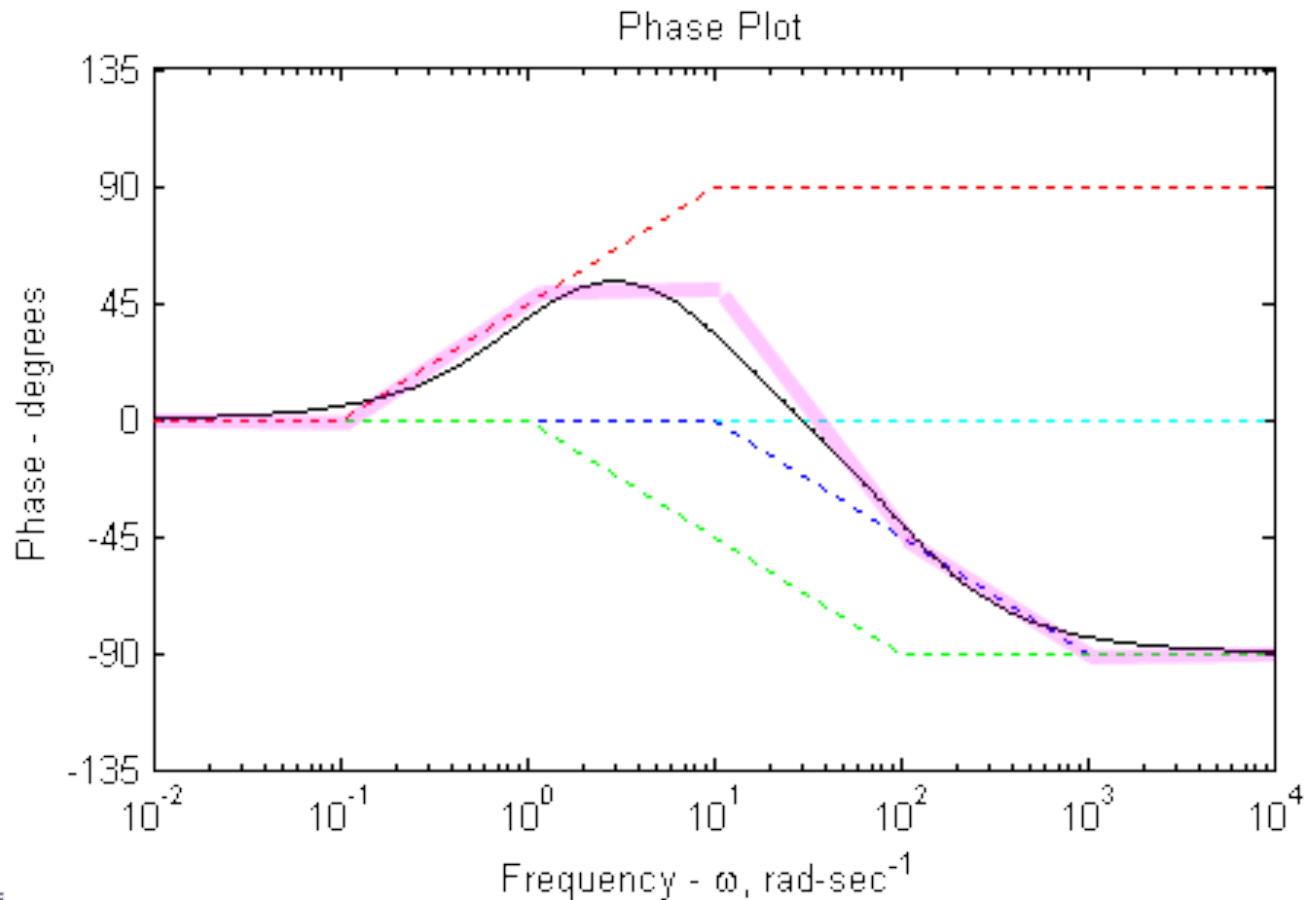
- **Step 3: Draw the Bode diagram for each part.**
- The **constant** is the **cyan line** (A quantity of 0.1 is equal to -20 dB). The phase is constant at 0 degrees.
- The **zero at 1 rad/sec** is the **red line**. It is 0 dB up to the **break frequency**, then **rises at 20 dB/dec**. The phase is 0 degrees up to 1/10 the break frequency (0.1 rad/sec) then rises linearly to 90 degrees at 10 times the break frequency (10 rad/sec).

Magnitude Plot



- The **pole at 10 rad/sec** is the **green line**. It is 0 dB up to the break frequency, then drops off with a **slope of -20 dB/dec**. The phase is 0 degrees up to **1/10 the break frequency** (1 rad/sec) then drops linearly down to -90 degrees at **10 times the break frequency** (100 rad/sec).
- The **pole at 100 rad/sec** is the **blue line**. It is 0 dB up to the **break frequency**, then drops off with a **slope of -20 dB/dec**. The phase is 0 degrees up to **1/10 the break frequency** (10 rad/sec) then drops linearly down to -90 degrees at **10 times the break frequency** (1000 rad/sec).

Step 4: Draw the overall Bode diagram by adding up the results from step 3.



Bode Diagram Example

- Bode diagram of

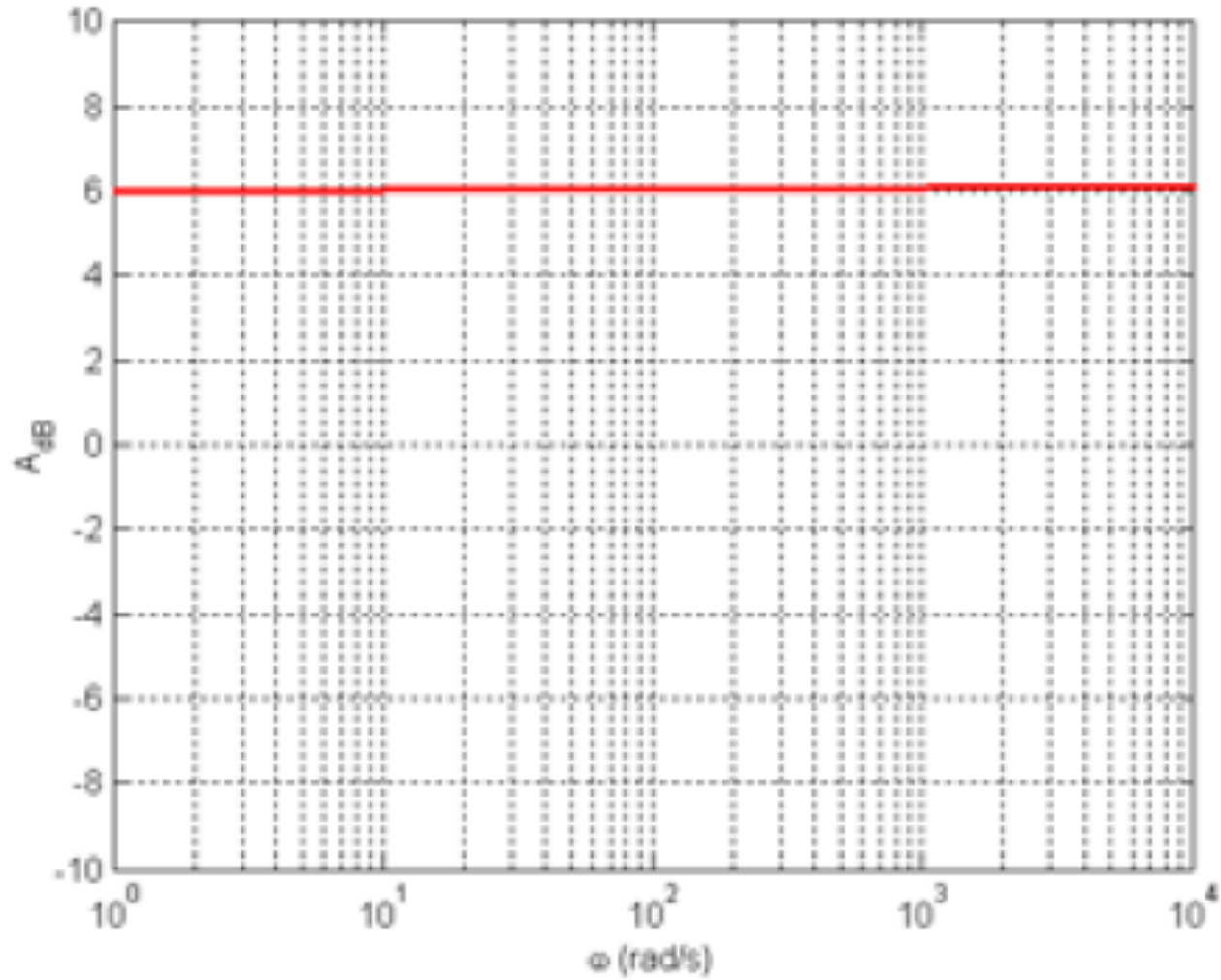
$$G = 2 \frac{(1 + 0.1s)}{s(1 + 0.01s)}$$

$$A_{dB} = 20 \log \left(2 \frac{1}{|j\omega|} \frac{|1 + 0.1j\omega|}{|1 + 0.01j\omega|} \right)$$

$$= 20 \log(2) + 20 \log|1 + 0.1j\omega|$$

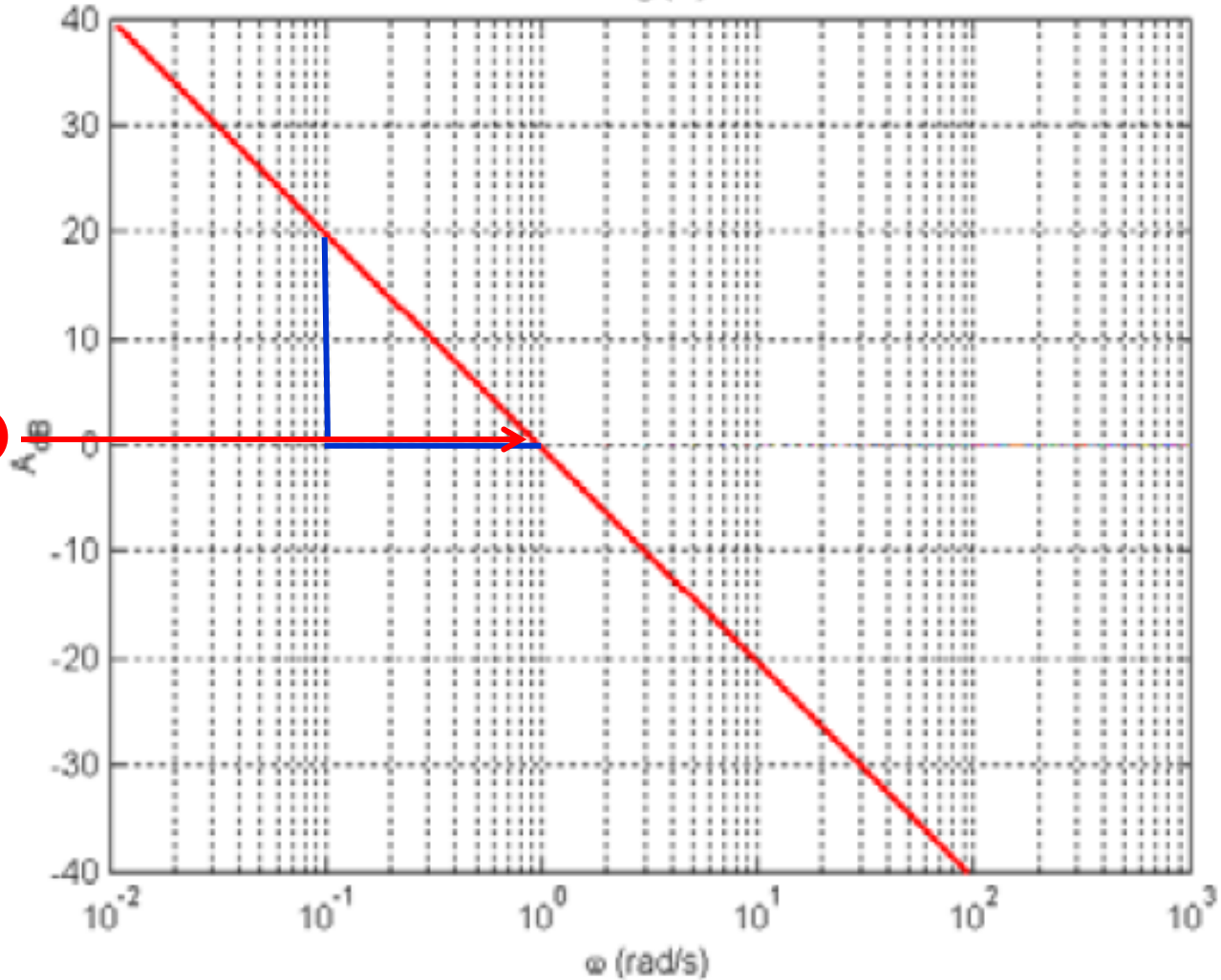
$$- 20 \log|j\omega| - 20 \log|1 + 0.01j\omega|$$

$$A_{dB} = 20 \log(2) = 6dB$$

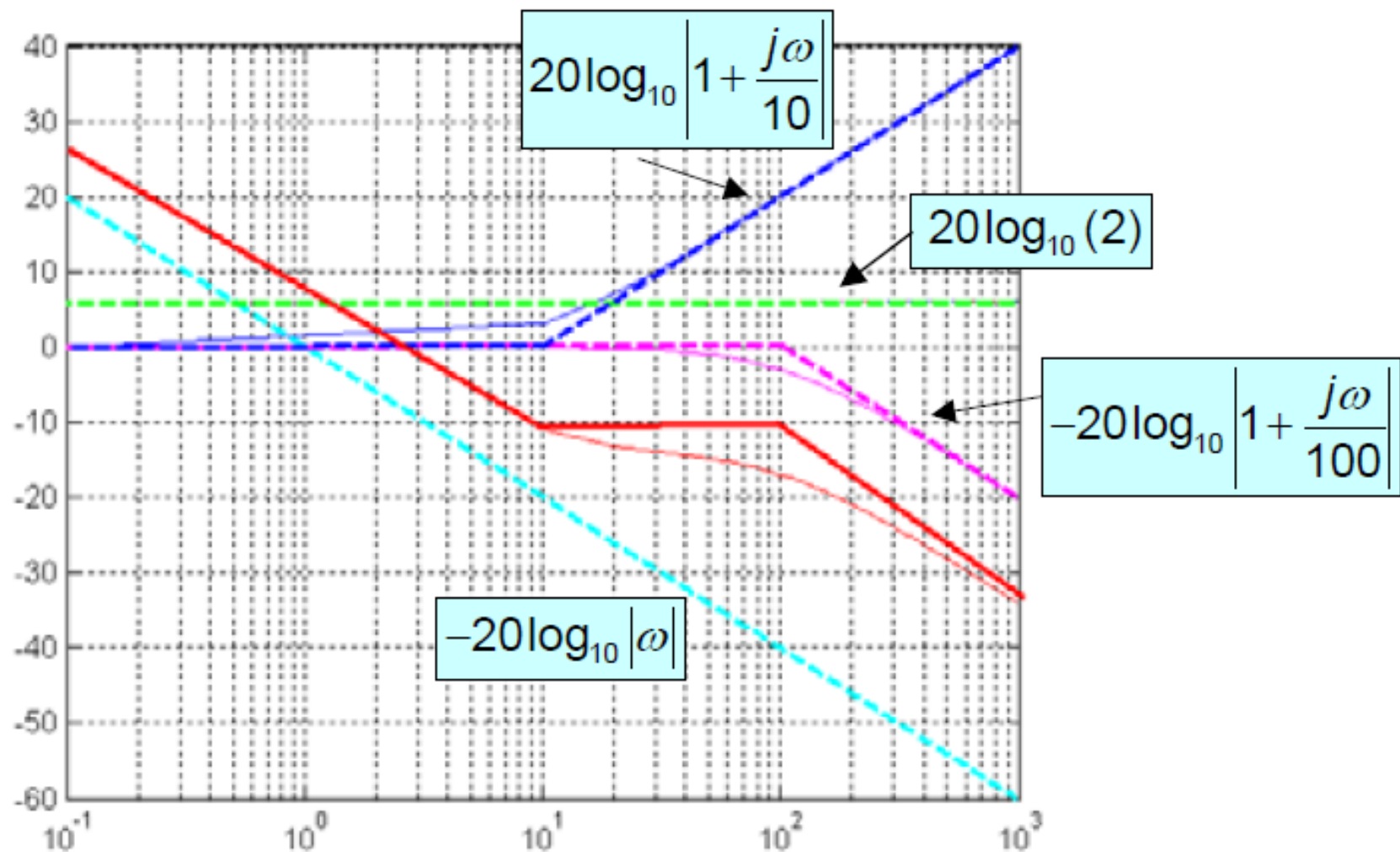


$$A_{dB} = -20 \log |j\omega|$$

$-20 \log (\omega)$

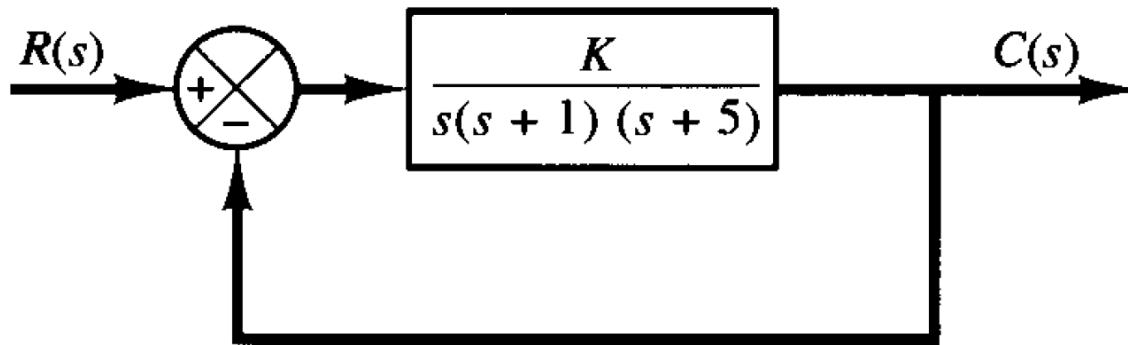


Slope=-20
pass (0,10⁰)



Stability Analysis

- The phase and gain margins can easily be obtained from the Bode diagram.



- For $K = 10$ and $K = 100$, find the phase and gain margins

