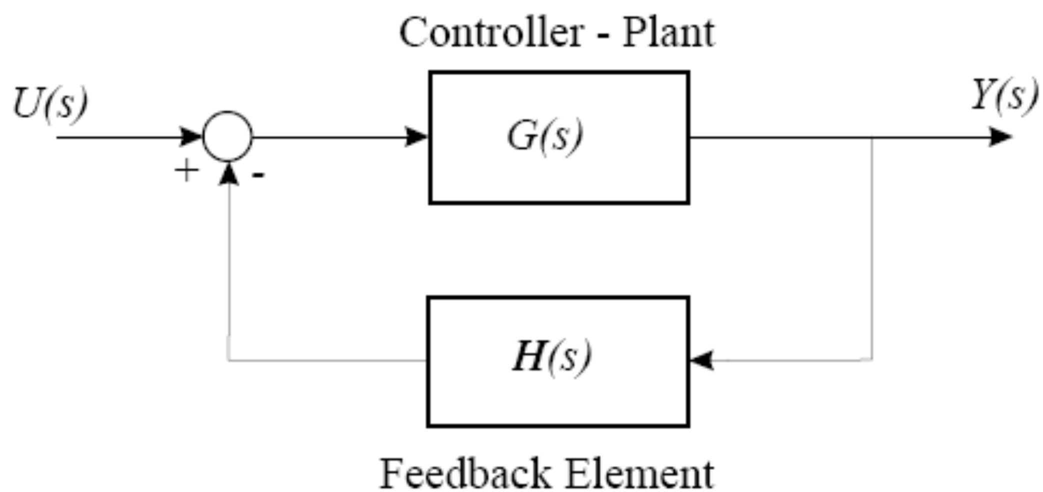
Root Locus Technique

CheEng@TongjiU
Min Huang, PhD



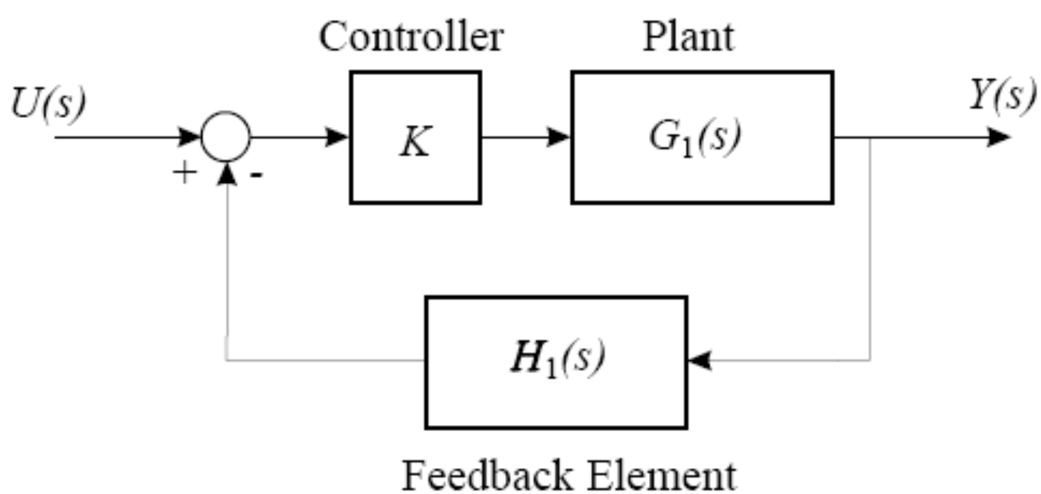
- The characteristic equation

$$1 + G(s)H(s) = 0$$

- Can also be write as

$$1 + KG_1(s)H_1(s) = 0$$

- where represents all static gains present in the loop



- The closed-loop transfer function of this system is given by

$$M(s) = \frac{Y(s)}{U(s)} = \frac{G(s)}{1 + G(s)H(s)} = \frac{G(s)}{1 + KG_1(s)H_1(s)}$$

$$= \frac{G(s)}{1 + \dfrac{K(s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0)}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}}$$

$$= \frac{G(s)(s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0)}{(s^n + a_{n-1}s^{n-1} + \cdots + a_0) + K(s^m + b_{m-1}s^{m-1} + \cdots + b_0)}$$

$$n \geq m$$

- The corresponding characteristic equation is

$$(s^n + a_{n-1}s^{n-1} + \cdots + a_0)$$

$$+K(s^m + b_{m-1}s^{m-1} + \cdots + b_0) = 0$$

- The question to be answered by the root locus technique is: what can be achieved by changing the static gain K, theoretically, from $-\infty$ to $+\infty$?

- answer to this question led to the development of the root locus technique. It was discovered by W. Ewans in 1948 and was mathematically formulated in 1950 in his famous paper (Ewans, 1950)
- The main idea behind the root locus technique is hidden in equation

$$G_1(s)H_1(s) = -\frac{1}{K}$$

What can you find ?

- It actually represents two equations (for real and imaginary parts
- Consider only the $0 \le K < \infty$ part, for the magnitudes

$$| G_1(s)H_1(s) | = \frac{1}{K}$$

- and for the phase angles

$$\measuredangle G_1(s)H_1(s) = (2l+1)\pi, \qquad l = 0, \pm 1, \pm 2, \ldots$$

- If we factor $G(s)_1H(s)_1$ as

$$G_1(s)H_1(s) = \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

- by using elementary algebra with complex numbers, we get

$$|\, G_1(s)H_1(s) \,| = \frac{\prod_{i=1}^{m} |\, s-z_i \,|}{\prod_{i=1}^{n} |\, s-p_i \,|} = \frac{1}{K}$$

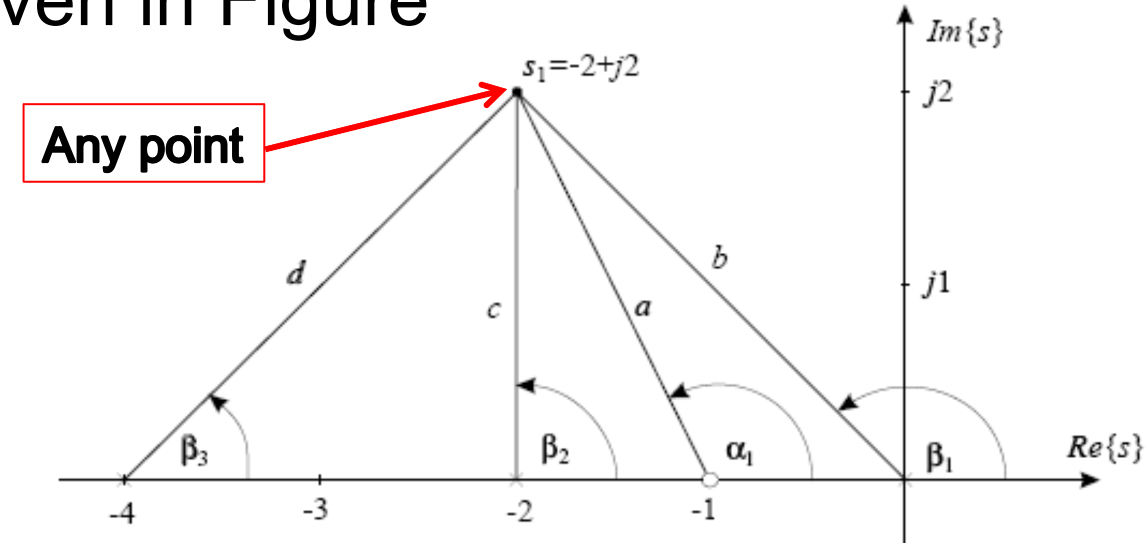
- And

$$\measuredangle G_1(s)H_1(s) = \sum_{i=1}^{m} \measuredangle(s - z_i) - \sum_{i=1}^{n} \measuredangle(s - p_i) = (2l + 1)\pi$$

- Example: the open-loop transfer function

$$G(s)H(s) = \frac{K(s+1)}{s(s+2)(s+4)}$$

- The locations of the open-loop poles and zeros are given in Figure



- Take any point (in the complex plane. If that point belongs to the root locus, it must satisfy both equations (for the magnitude and for the phase)—Criteria!

- For example, for the point $s_1 = -2 + j2$

$$\frac{|\,s_1 + 1\,|}{|\,s_1 + 0\,||\,s_1 + 2\,||\,s_1 + 4\,|} = \frac{1}{K} = \frac{a}{b \cdot c \cdot d}$$

$$= \frac{\sqrt{5}}{\sqrt{8} \cdot 2 \cdot \sqrt{8}} \Rightarrow K = \frac{16}{\sqrt{5}}$$

- Thus, if the point s1(belongs to the root locus, the static gain K at that point must be equal to $16/\sqrt{5}$.

- It follows that for the point s1 (the following must be satisfied)

$$\measuredangle G(s_1)H(s_1) = \measuredangle(s_1+1) - \measuredangle(s_1+0) - \measuredangle(s_1+2) - \measuredangle(s_1+4)$$

$$= (2l+1)\pi$$

- Which means

$$\alpha_1 - \beta_1 - \beta_2 - \beta_3 = (2l+1)\pi, \qquad l = 0, \pm 1, \pm 2, \ldots$$

$$116.57^\circ - 135^\circ - 90^\circ - 45^\circ = -143.33^\circ \neq (2l+1)\pi$$

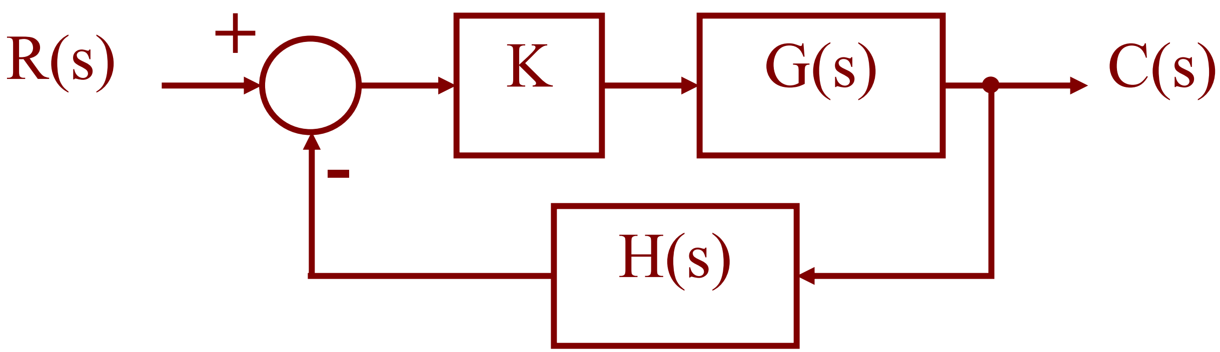
$$\text{for any } l$$

- We can conclude that the point cannot belong to the root locus since the phase equation is apparently not valid !

Rules for Making Root Locus Plots

- The closed loop transfer function of the system shown is

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)}$$



- So the characteristic equation (*c.e.*) is

$$1 + KG(s)H(s) = 1 + K\frac{N(s)}{D(s)} = 0$$

- or

$$D(s) + K\,N(s) = 0$$

- As K changes, so do locations of closed loop poles (i.e., zeros of *c.e.*). The table below gives rules for sketching the location of these poles for K=0→∞ (i.e., K≥0)

- **Definitions**
  - The loop gain is KG(s)H(s) or .
  - N(s), the numerator, is an *m*th order polynomial; D(s), is *n*th order.
  - N(s) has zeros at z_i (i=1..*m*); D(s) has them at p_i (i=1..*n*).
  - The difference between *n* and *m* is *q*, so $q=n-m$. ($q \geq 0$)

- **1. Symmetry**
  - The locus is symmetric about real axis (i.e., complex poles appear as conjugate pairs)
- **2. Number of Branches**
  - There are *n* branches of the locus, one for each closed loop pole.
- **3. Starting and Ending Points**
  - The locus starts (K=0) at poles of loop gain, and ends ($K \to \infty$) at zeros. Note: this means that there will be *q* roots that will go to infinity as $K \to \infty$.

- **4. Locus on Real Axis**
  - The locus exists on real axis to the left of an odd number of poles and zeros
- **5. Angles of Asymptotes**

$$\alpha_m = \pm \frac{(2m + 1)\pi}{p - z}$$

$$m = 0,1,2, \ldots, m = p - z$$

- **6. Intersection of Asymptotes**

$$s_r = \frac{\sum_{poles} - \sum_{zeros}}{p - z}$$

- **7. Break-Away /-In Points on Real Axis**

$$1 + G(s)H(s) = \frac{K}{s(s + p_1)(s + p_2)\cdots}$$

$$\frac{dK(\sigma)}{d\sigma} = 0$$

Solve for σ.

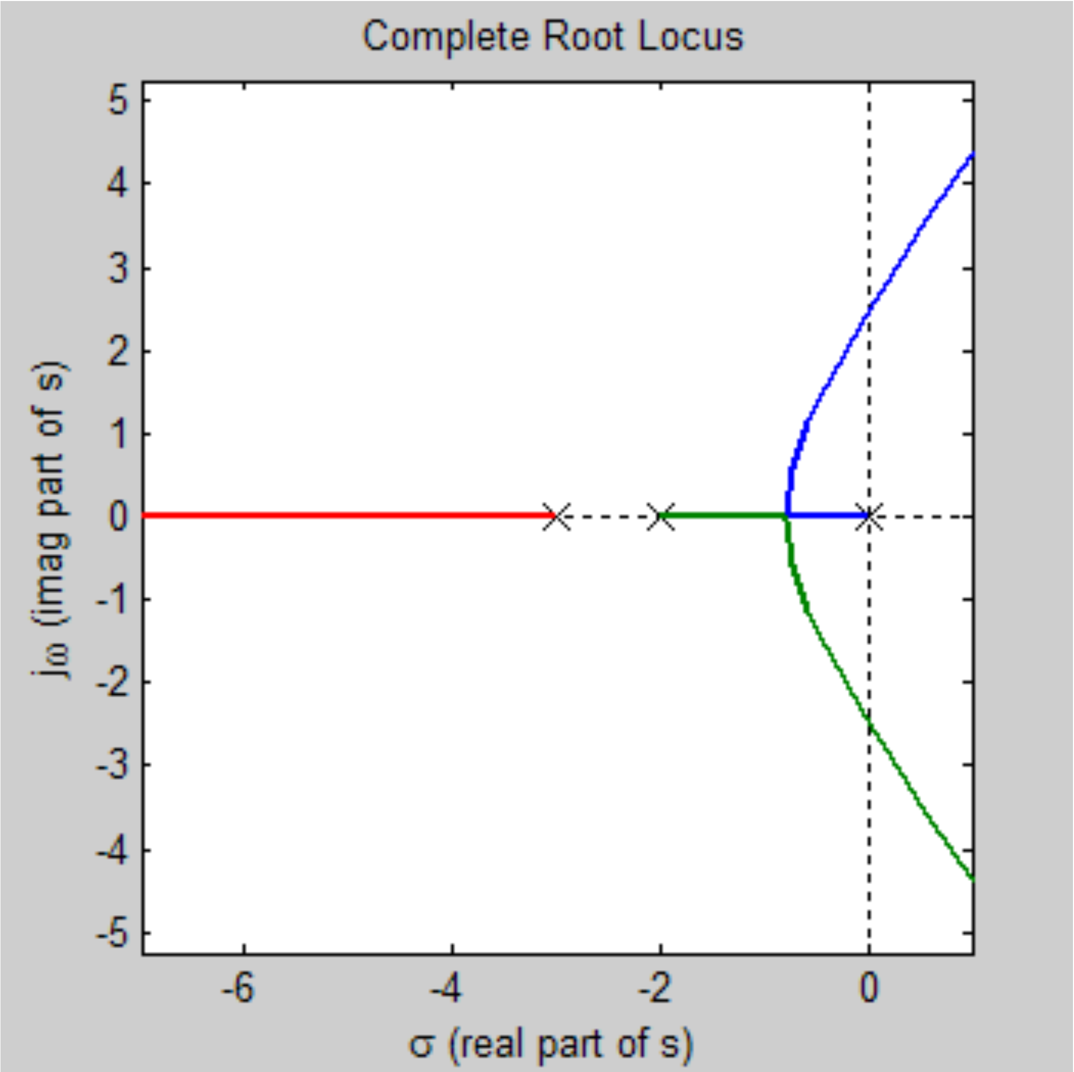
- **8. Locus Crosses Imaginary Axis**
  - Use Routh-Hurwitz to determine where the locus crosses the imaginary axis
- **9. Given Gain "K," Find Poles**
  - Rewrite *c.e.* as D(s)+KN(s)=0. Put value of K into equation, and find roots of *c.e.*.

(Symbol Algebra)

Example 1

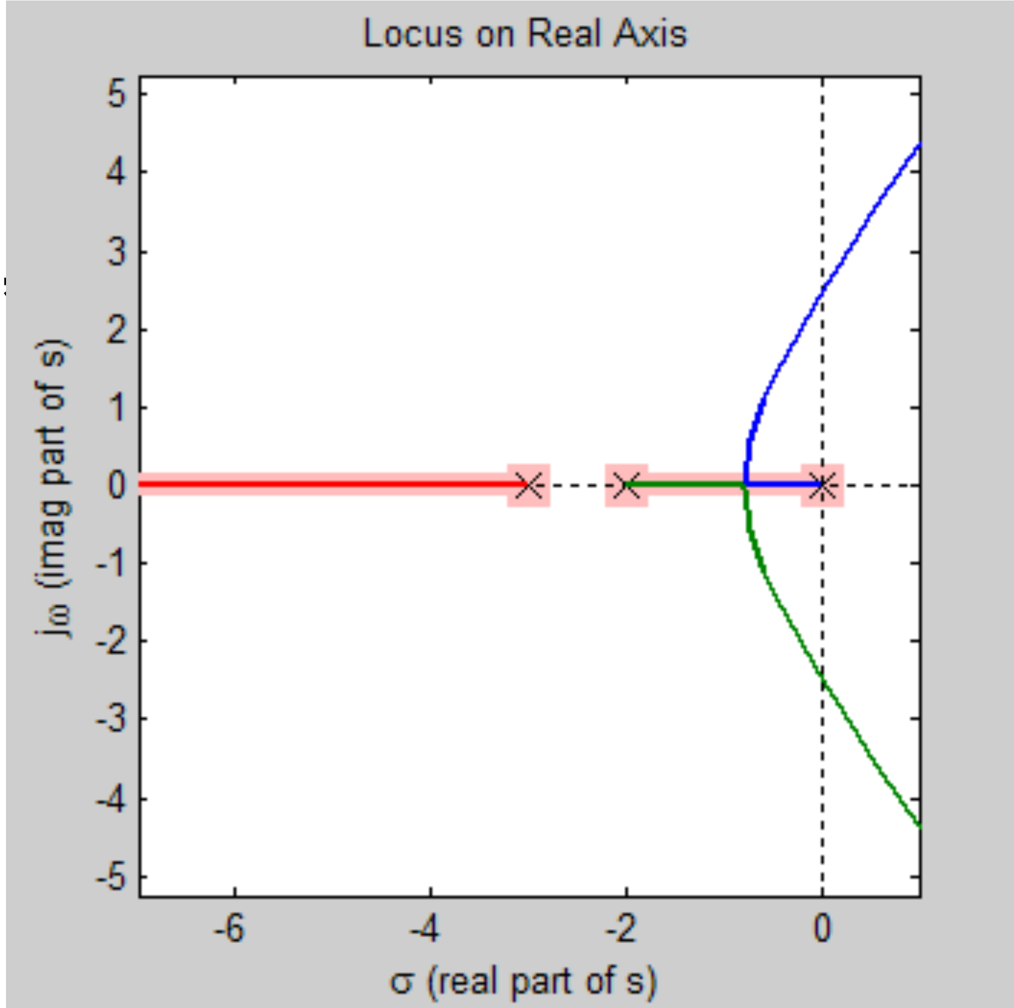
- Close loop transfer function

$$G(s)H(s)=\frac{1}{s(s^2+5s+6)}$$



- Root Locus Symmetry
  - As you can see, the locus is symmetric about the real axis
- Number of Branches
  - The open loop transfer function, G(s)H(s), has 3 poles, therefore the locus has 3 branches. Each branch is displayed in a different color.
- Start/End Points
  - We have p=3 finite poles, and z=0 finite zeros
  - we also have m=p-z=3 zeros at infinity

- Locus on Real Axis
- on the real axis, we have 3 poles at s = -2, -3, 0, and we have no zeros
- Root locus exists on real axis between:
  0 and -2
  -3 and negative infinity



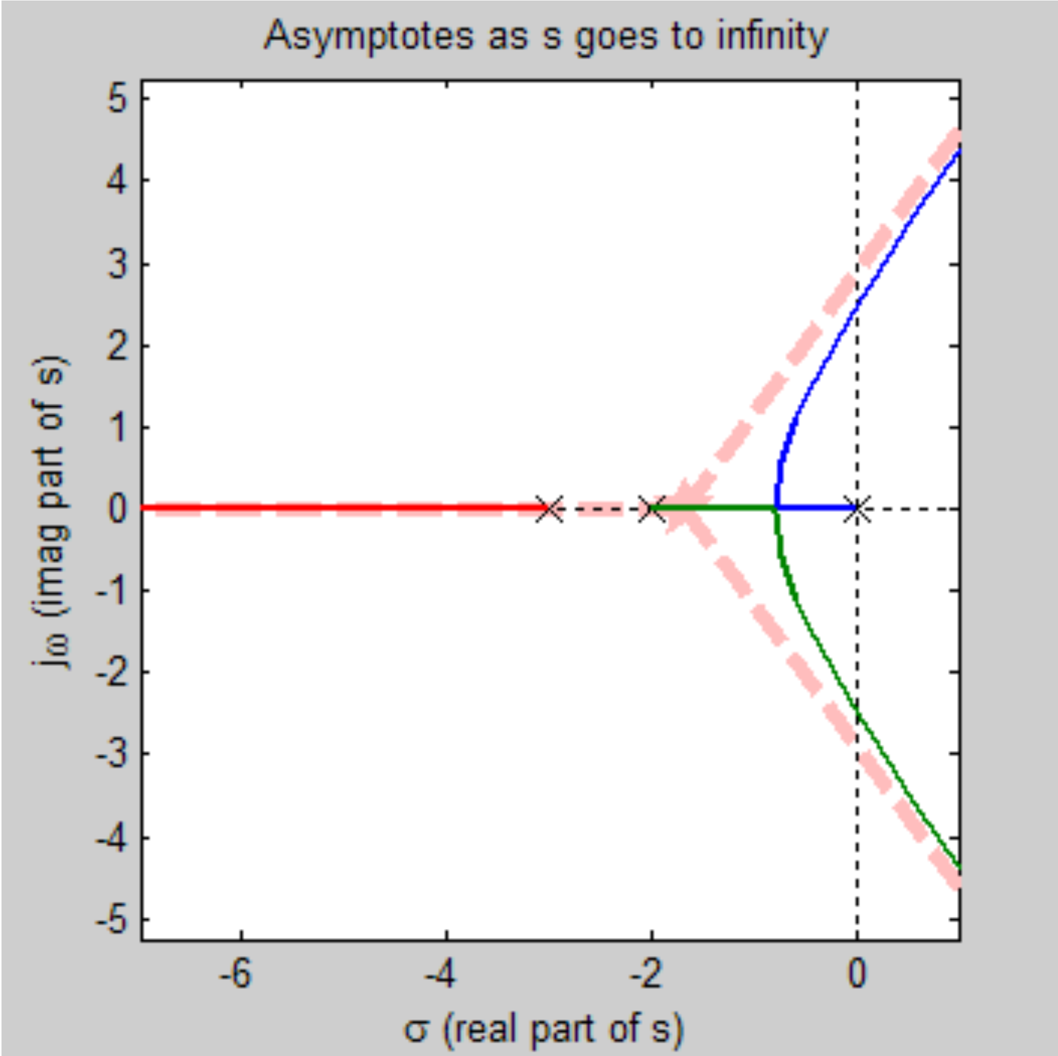
- $\alpha_m = \pm \frac{(2m+1)\pi}{p-z}$

p = # of poles

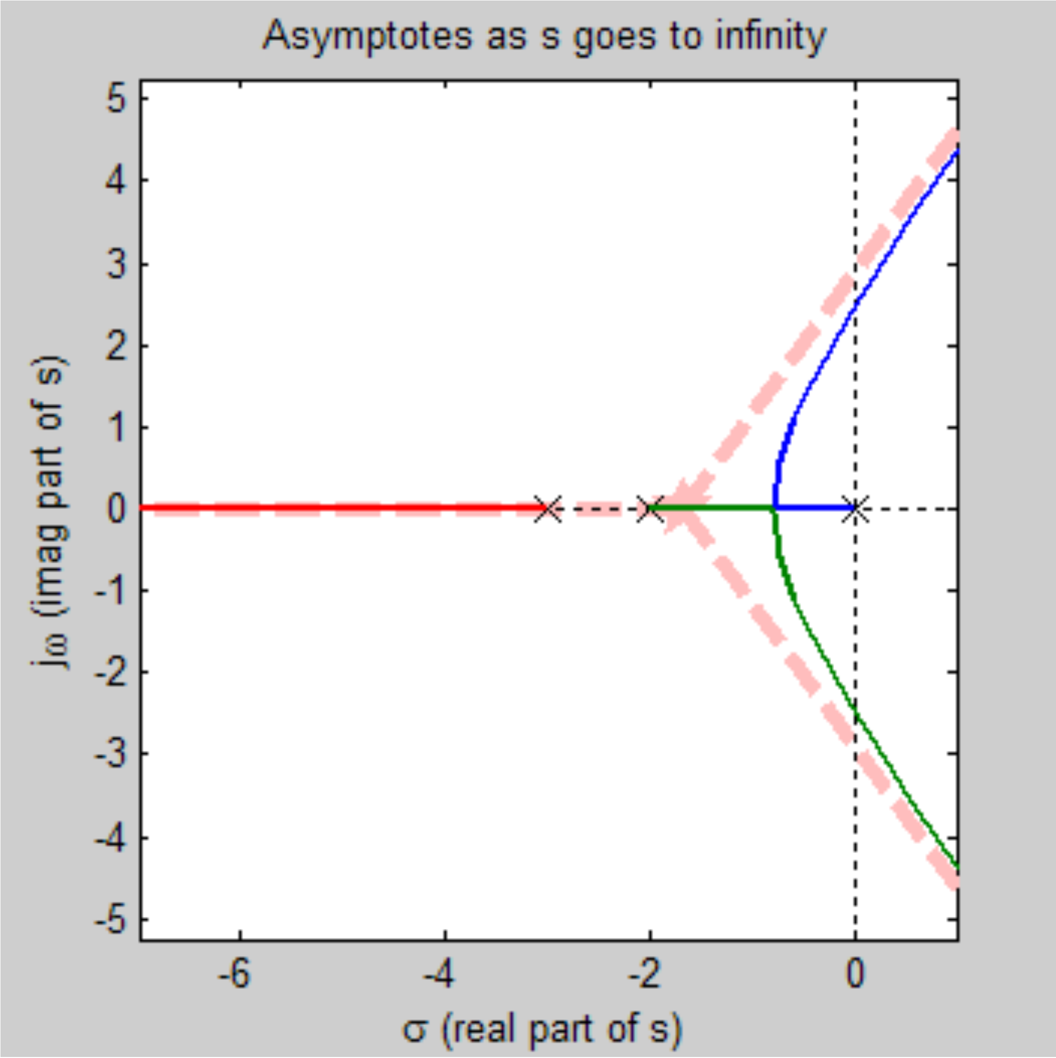
z = # of zeros

m = 0,1,2,… m=p-z

- **p=3 finite poles, and z=0 finite zeros, therefore we have m=p-z=3 zeros at infinity**
- **$\pm 60^\circ$ + 180°**



- **There exists 3 poles at 0, -3, -2, ...so sum of poles=-5.**
  There exists 0 zeros, ...so sum of zeros=0.
- **s_r = ((sum of poles)-(sum of zeros))/(p-z)**
- **= $\frac{-5}{3}$ = -1.67.**
  Intersect is at -1.67



$$G(s)H(s) = \frac{K}{s(s^2+5s+6)}$$

$$1 + G(s)H(s) = 1 + \frac{K}{s(s^2+5s+6)} = 0$$

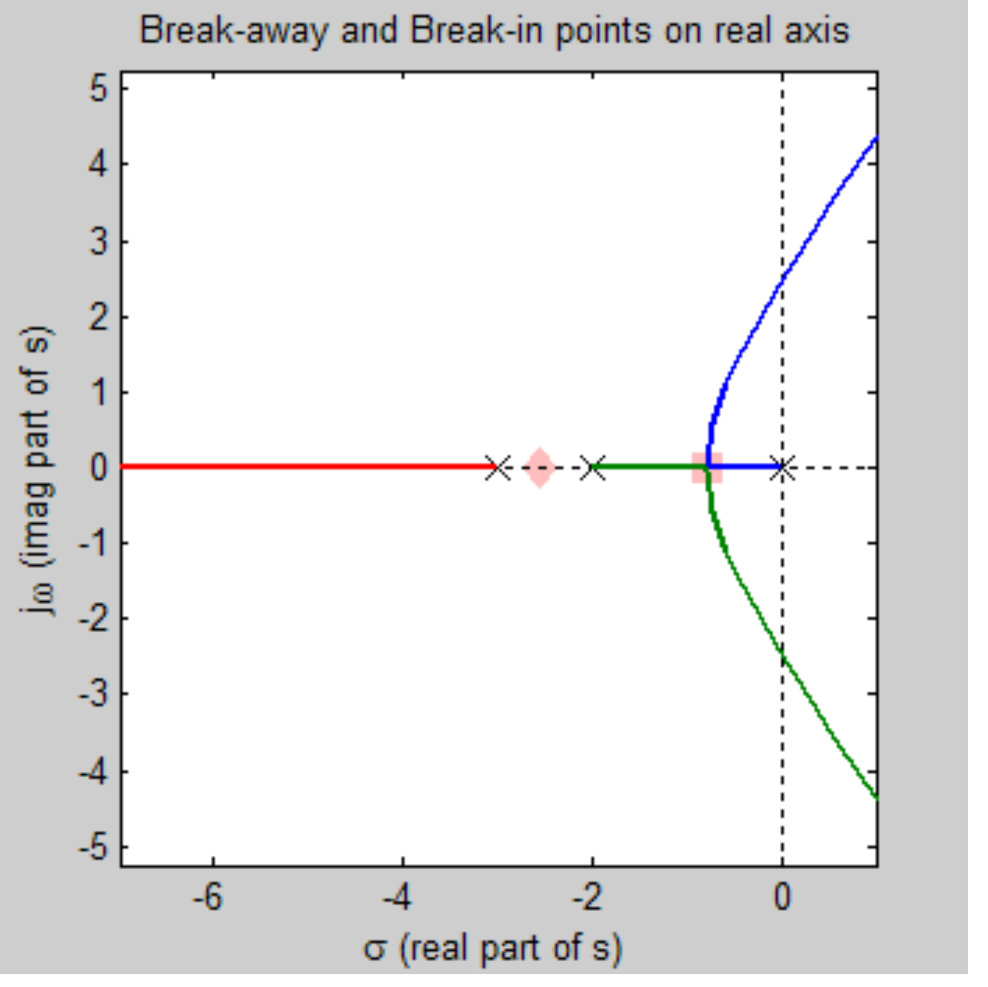
$$K = -\, s(s^2 + 5s + 6)$$

$$K(\sigma) = -\, (\sigma^3 + 5\sigma^2 + 6\sigma)$$

$$\frac{dK(\sigma)}{d\sigma} = 0$$

$$0 = -\, (3\sigma^2 + 10\sigma + 6)$$

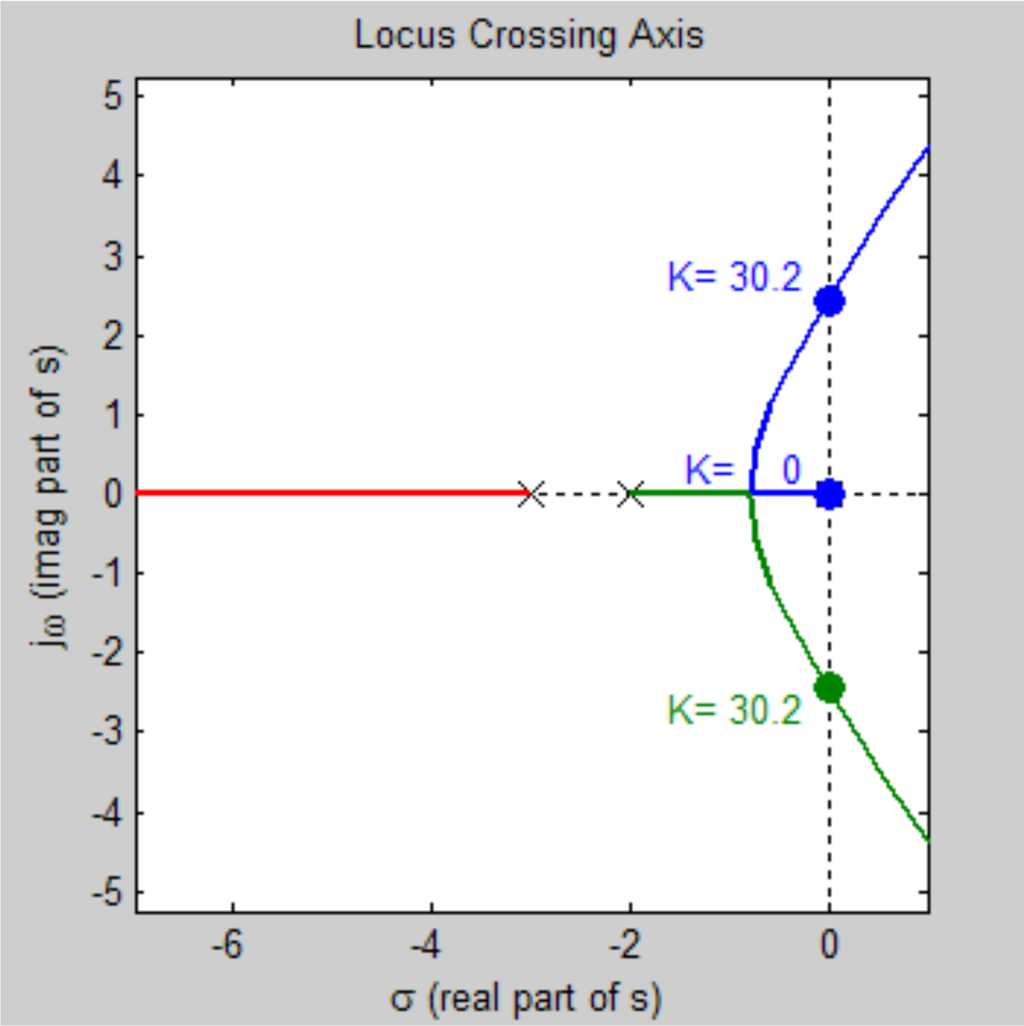
- **Break-Out and In Points on Real Axis**
- **$3\ \sigma^2 + 10\ \sigma + 6 = 0$,**
- **$\sigma = -2.5, -0.78$**
- **root at $\sigma = -0.78$ on the locus (i.e., K>0). Break-away (or break-in) points on the locus are shown by squares.**
  break-away (or break-in) with K less than 0 are shown with diamonds



$$s(s^2 + 5s + 6) + K = 0$$

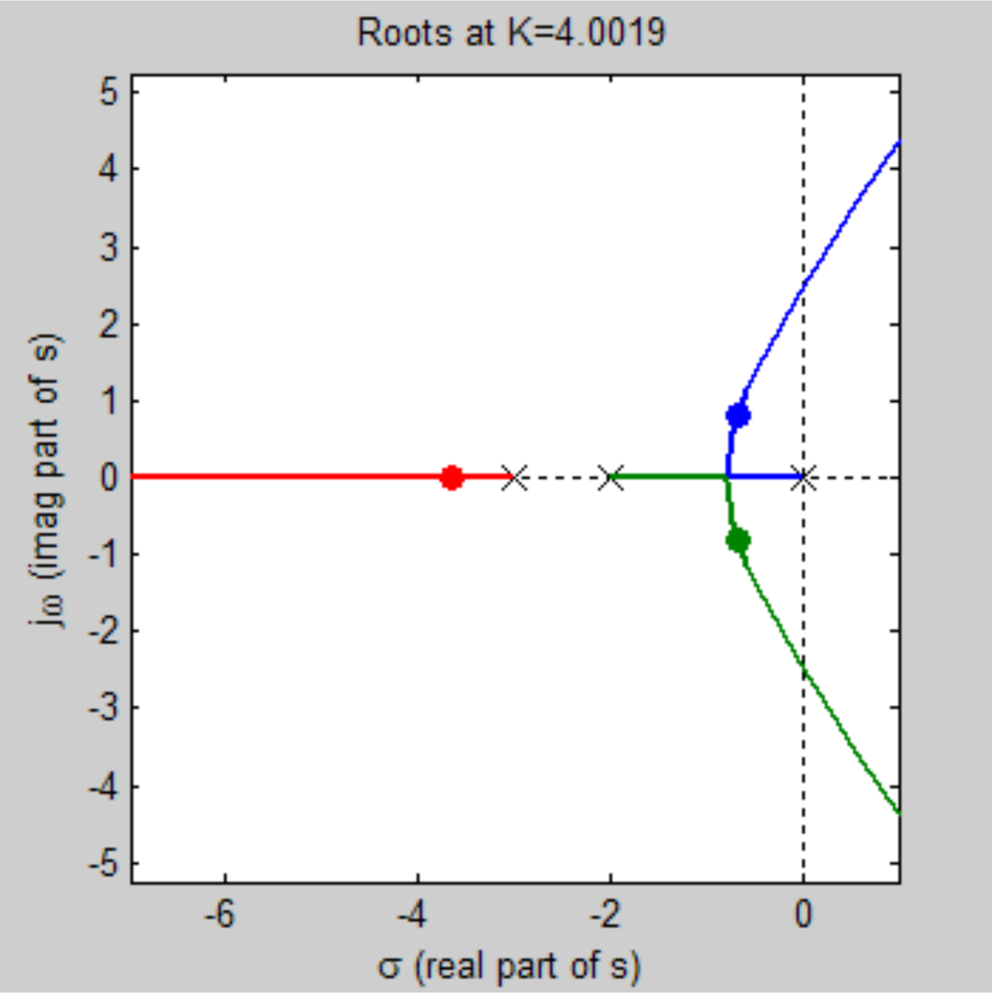
s^3	1	6
s^2	5	K
s^1	$\frac{30-K}{5}$	
s^0	K	

K = 30, s = 0 ± 2.45j



Changing K Changes Closed Loop Poles

- For example with K=4.00188,
- s3 + 5 s2 + 6 s + 4.0019 = 0
- s = -3.7, -0.67 ± 0.8j



Choose Pole Location and Find K

- K = -D(s)/N(s) = -(s3 + 5 s2 + 6 s) / (1)
- choose s= -0.7 + 0.84j
- D(s)=-4.15 + -0.222j, N(s)= 1 + 0j
- K=-D(s)/N(s)= 4.15 + 0.222j, is not exactly on the locus, pick real part of K (4.15)
- For this K there exist 3 closed loop poles at s = -3.7, -0.66±0.83j.