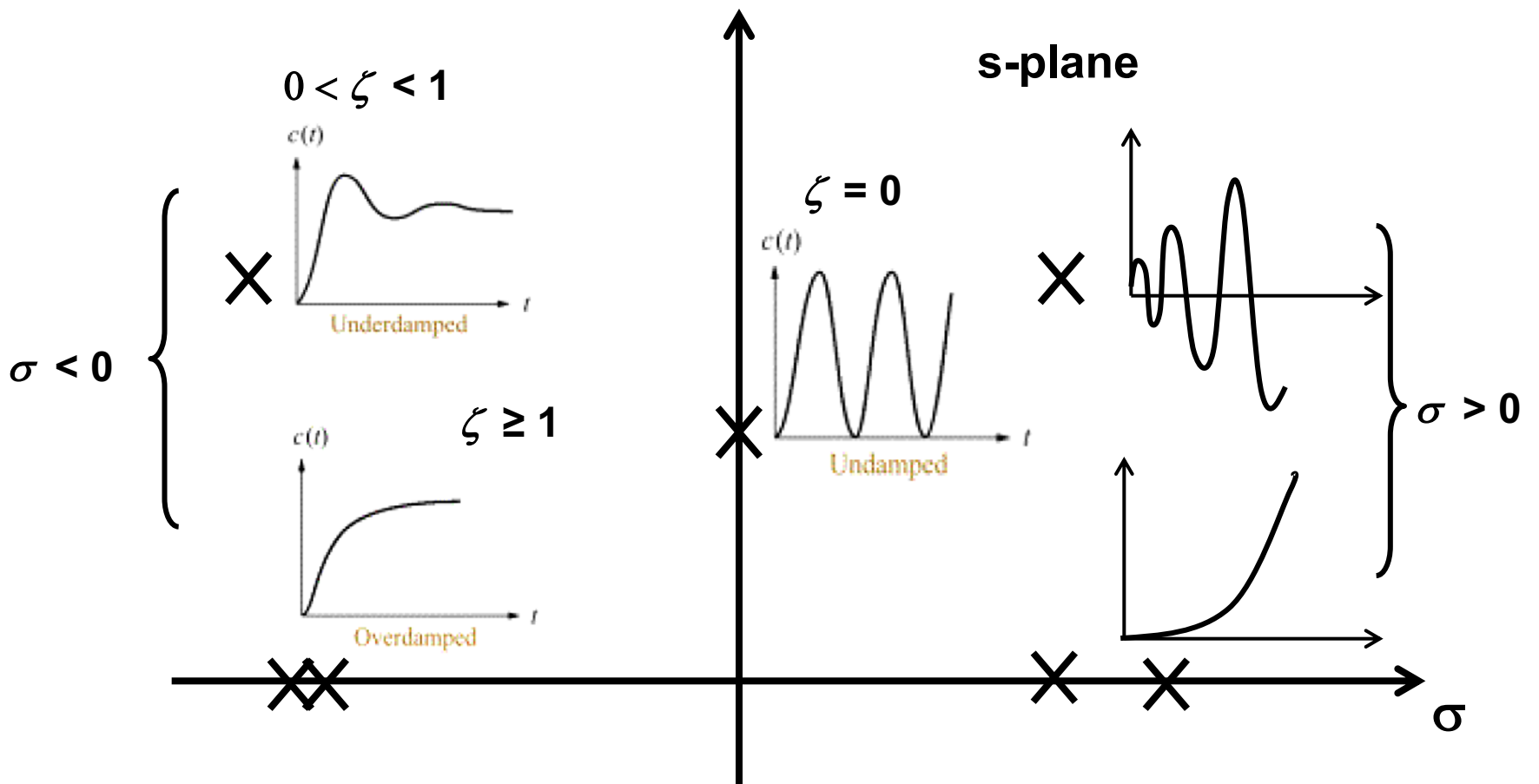


Performance Criteria

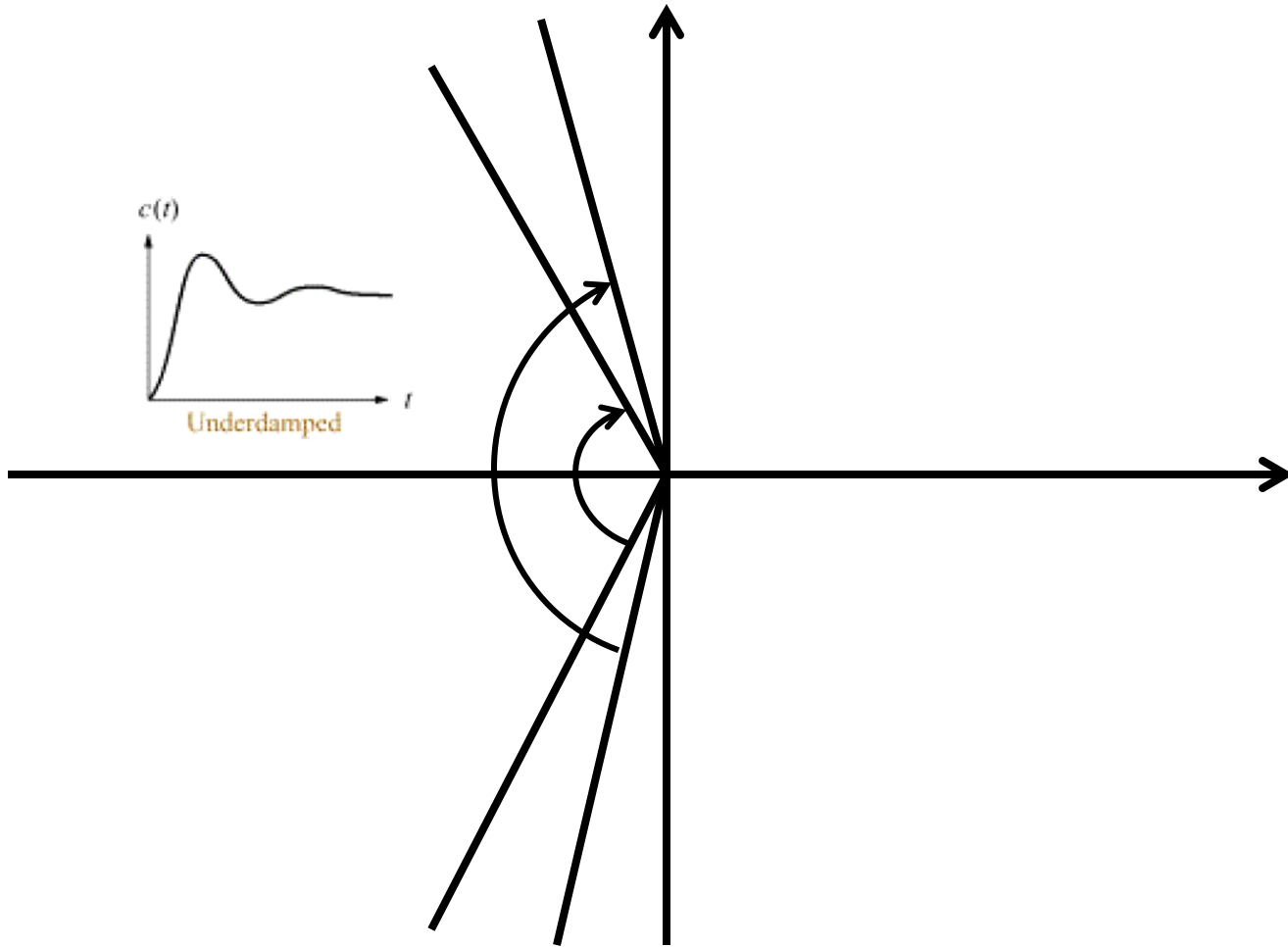
Performance of a Feedback Control System

- **In the early days**
 - **Stability** ✓
 - **Static Accuracy** ✓
- **Modern Complex Control System Also Interested in**
 - **Sensitivity** ✓
 - **Transient Response** ✓
 - **Residual Noise Jitter**

Stability



Stability



Sensitivity

- **Definition**

$$S_K^H(s) = \frac{d \ln H(s)}{d \ln K(s)} = \frac{\% \text{ change in } H(s)}{\% \text{ change in } K(s)}$$

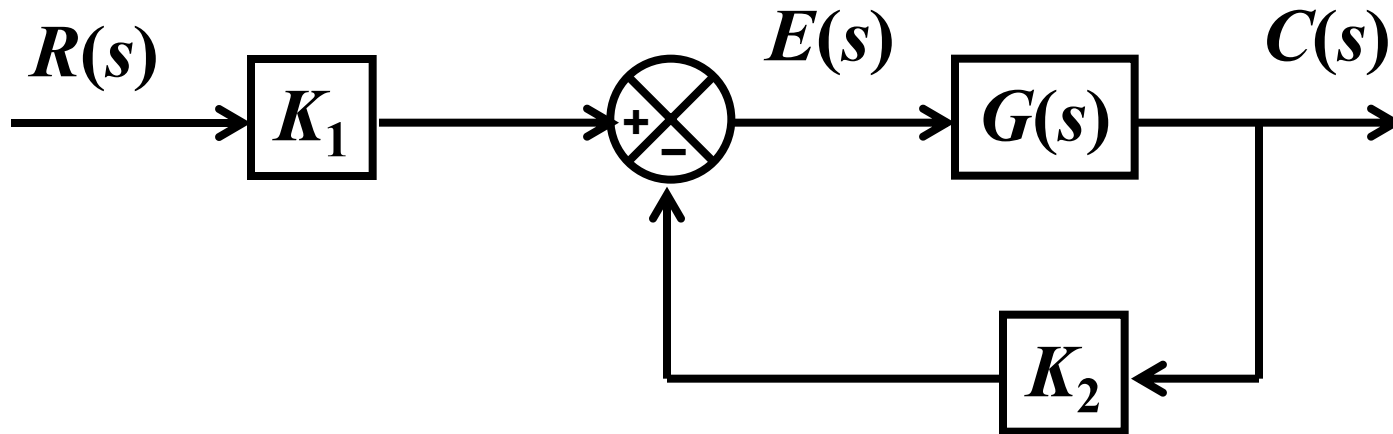
where

$$H(s) = \frac{C(s)}{R(s)}$$

- **Rearrange**

$$S_K^H(s) = \frac{dH(s) / H(s)}{dK(s) / K(s)}$$

Example



$$H(s) = \frac{K_1(s)G(s)}{1 + K_2(s)G(s)}$$

Example

- **Sensitivity of $H(s)$ with respect to K_1**

$$S_{K_1}^H(s) = \frac{dH(s)/H(s)}{dK_1(s)/K_1(s)} = \frac{K_1(s)}{H(s)} \frac{dH(s)}{dK_1(s)}$$

where

$$\frac{dH(s)}{dK_1(s)} = \frac{G(s)}{1 + K_2(s)G(s)} = \frac{H(s)}{K_1(s)}$$

Therefore,

$$S_{K_1}^H(s) = \frac{dH(s)/H(s)}{dK_1(s)/K_1(s)} = 1$$

Example

- **Sensitivity of $H(s)$ with respect to K_2**

$$S_{K_2}^H(s) = \frac{dH(s)/H(s)}{dK_2(s)/K_2(s)} = \frac{K_2(s)}{H(s)} \frac{dH(s)}{dK_2(s)}$$

– where

$$\begin{aligned} \frac{dH(s)}{dK_2(s)} &= \frac{0 - K_1 G^2(s)}{[1 + K_2(s)G(s)]^2} \\ &= \frac{-K_2(s)G^2(s)}{K_1(s)[1 + K_2(s)G(s)]^2} \end{aligned}$$

Example

– Therefore

$$\begin{aligned} S_{K_2}^H(s) &= \frac{K_2(s)}{H(s)} \frac{-K_1^2 G^2(s)}{K_1(s)[1 + K_2(s)G(s)]^2} \\ &= \frac{-K_2(s)}{H(s)} \frac{H^2(s)}{K_1(s)} = \frac{-K_2(s)G(s)}{1 + K_2(s)G(s)} \end{aligned}$$

– When $K_2 G(s) \gg 1$

$$S_{K_2}^H(s) = \frac{-K_2(s)G(s)}{1 + K_2(s)G(s)} \approx -1$$

Example

- **Sensitivity of $H(s)$ with respect to $G(s)$**

$$S_G^H(s) = \frac{dH(s)/H(s)}{dG(s)/G(s)} = \frac{G(s)}{H(s)} \frac{dH(s)}{dG(s)}$$

where

$$\begin{aligned} \frac{dH(s)}{dG(s)} &= \frac{[1 + K_2(s)G(s)]K_1 - K_1(s)G(s)K_2(s)}{[1 + K_2(s)G(s)]^2} \\ &= \frac{K_1(s)}{[1 + K_2(s)G(s)]^2} \end{aligned}$$

Example

- **Therefore**

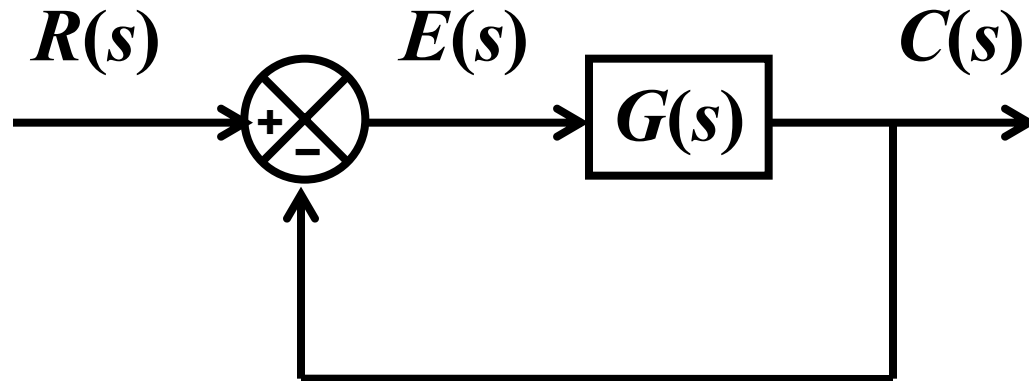
$$\begin{aligned} S_{G(s)}^H(s) &= \frac{G(s)}{H(s)} \frac{K_1(s)}{[1 + K_2(s)G(s)]^2} \\ &= \frac{1}{1 + K_2(s)G(s)} \end{aligned}$$

- **Desire to have**

$$1 + K_2(s)G(s) \gg 1$$

Quiz IX

Static Accuracy



$$\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)}$$

$$E(s) = R(s) \frac{1}{1 + G(s)}$$



$$1 - \frac{C(s)}{R(s)} = \frac{R - C}{R} = \frac{E}{R}$$

$$E(s) = R(s) \left(1 - \frac{C(s)}{R(s)} \right)$$

Static Accuracy

- **Apply the final-value theorem of the Laplace transform,**

$$e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} s \cdot R(s) \left(1 - \frac{C(s)}{R(s)} \right)$$

- **We usually interested in,**
 - **Position input**
 - **Velocity input**
 - **Acceleration input**

Static Accuracy

- **These inputs are usually called**

- **Unit step**

$$r(t) = U(t) \quad R(s) = 1/s$$

- **Unit ramp**

$$r(t) = tU(t) \quad R(s) = 1/s^2$$

- **Paraboloid**

$$r(t) = \frac{1}{2}t^2U(t) \quad R(s) = 1/s^3$$

Static Accuracy

- Assume the loop transfer function $G(s)$,

$$G(s) = \frac{K(1 + T_1s)(1 + T_2s)\cdots(1 + T_ms)}{s^n \left[(T_0s)^2 + 2\zeta\omega_n s + 1 \right] (1 + T_as)(1 + T_bs)\cdots(1 + T_qs)}$$

- Based on the pure integrations in the denominator of the open loop transfer function, the system is defined as the n^{th} type system

Static Accuracy

- **As the system type is increased, the accuracy is improved. However,**
- **in practice, we usually never design a control system greater than type 2 because it is very difficult to stabilize a control system containing more than two pure integrations.**
- **There will be a tradeoff between them**

Static Accuracy

- **Unit step**

$$e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} s \cdot R(s) \left(1 - \frac{C(s)}{R(s)} \right)$$

$$e_{ss} = \lim_{s \rightarrow 0} \frac{s(1/s)}{1 + G(s)} = \frac{1}{1 + \lim_{s \rightarrow 0} G(s)}$$

- **Define position constant**

Imply H = 1, that needs not have to

$$K_p = \lim_{s \rightarrow 0} G(s)$$

- **therefore**

$$e_{ss} = \frac{1}{1 + K_p}$$

Static Accuracy

- **Unit ramp**

$$e_{ss} = \lim_{s \rightarrow 0} \frac{s(1/s^2)}{1 + G(s)} = \frac{1}{\lim_{s \rightarrow 0} sG(s)}$$

- **Define velocity constant**

$$K_v = \lim_{s \rightarrow 0} sG(s)$$

- **therefore**

$$e_{ss} = \frac{1}{K_v}$$

Static Accuracy

- **Parabolic**

$$e_{ss} = \lim_{s \rightarrow 0} \frac{s(1/s^3)}{1 + G(s)} = \frac{1}{\lim_{s \rightarrow 0} s^2 G(s)}$$

- **Define acceleration constant**

$$K_a = \lim_{s \rightarrow 0} s^2 G(s)$$

- **therefore**

$$e_{ss} = \frac{1}{K_a}$$

Static Accuracy

- If the input composed of all three forms,

$$r(t) = AU(t) + BtU(t) + \frac{1}{2}Ct^2U(t)$$

- No pure integration

$$e_{ss} = \frac{1}{1 + K_p} + \infty + \infty$$

Static Accuracy

– One pure integration

$$e_{ss} = 0 + \frac{B}{K_v} + \infty$$

– Two pure integration

$$e_{ss} = 0 + 0 + \frac{C}{K_a}$$

Static Accuracy

- **Relationship between static error constants to closed-loop poles and zeros**

– A close loop transfer function,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

– The relation between the input and error,

$$\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)}$$

Imply H = 1, that
needs not have to

Static Accuracy

– And

$$E(s) = R(s) - C(s)$$

– It is evident that

$$\frac{C(s)}{R(s)} = 1 - \frac{E(s)}{R(s)}$$

– Assume $C(s)/R(s)$ is as

$$\frac{C(s)}{R(s)} = \frac{K(s + z_1)(s + z_2) \cdots (s + z_m)}{(s + p_1)(s + p_2) \cdots (s + p_n)} = \frac{K \prod_{j=1}^m (s + z_j)}{\prod_{j=1}^n (s + p_j)}$$

Static Accuracy

- And expand $E(s)/R(s)$ as power series in s

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)} = \frac{1}{1+K_p} + \frac{1}{K_v} s + \frac{1}{K_a} s^2 + \dots$$

- **Position Constant**

- Let s approach 0,

$$\lim_{s \rightarrow 0} \frac{E(s)}{R(s)} = \lim_{s \rightarrow 0} \frac{1}{1+G(s)} = \frac{1}{1+K_p}$$

- therefore $\frac{C(0)}{R(0)} = 1 - \frac{E(0)}{R(0)} = \frac{K_p}{1+K_p}$

Static Accuracy

– Solving for K_p in terms of $C(s)/R(s)$

$$K_p = \frac{C(0)/R(0)}{1 - C(0)/R(0)}$$

$$\frac{C(0)}{R(0)} = \frac{K(0 + z_1)(0 + z_2)\cdots(0 + z_m)}{(0 + p_1)(0 + p_2)\cdots(0 + p_n)} = \frac{K \prod_{j=1}^m z_j}{\prod_{j=1}^n p_j}$$

– therefore

$$K_p = \frac{K \prod_{j=1}^m z_j}{\prod_{j=1}^n p_j - K \prod_{j=1}^m z_j}$$

Static Accuracy

- **Velocity Constant**

$$\frac{C(s)}{R(s)} = 1 - \frac{E(s)}{R(s)}$$

– **And**

$$\frac{C(s)}{R(s)} = 1 - \frac{E(s)}{R(s)} = 1 - \left(\frac{1}{1 + K_p} + \frac{1}{K_v} s + \frac{1}{K_a} s^2 + \dots \right)$$

– **Take derivative with respect to s**

$$\frac{d}{ds} \left(\frac{C(s)}{R(s)} \right) = - \frac{1}{K_v}$$

Static Accuracy

– Therefore,

$$\frac{1}{K_v} = - \frac{\left[\frac{d}{ds} \left(\frac{C(s)}{R(s)} \right) \right]_{s=0}}{\left[\frac{C(s)}{R(s)} \right]_{s=0}} = - \left[\frac{d}{ds} \ln \frac{C(s)}{R(s)} \right]_{s=0}$$

– and

$$\frac{1}{K_v} = - \left\{ \frac{d}{ds} \left[\ln K + \ln(s + z_1) + \cdots + \ln(s + z_m) - \ln K - \ln(s + p_1) - \cdots - \ln(s + p_n) \right] \right\}_{s=0}$$

Static Accuracy

– It may be simplified as

$$\frac{1}{K_v} = - \left(\frac{1}{s + z_1} + \dots + \frac{1}{s + z_m} - \frac{1}{s + p_1} - \dots - \frac{1}{s + p_n} \right)_{s=0}$$

– or

$$\frac{1}{K_v} = \sum_{j=1}^n \frac{1}{p_j} - \sum_{j=1}^m \frac{1}{z_j}$$

Static Accuracy

- **Acceleration Constant**

$$\frac{d^2}{ds^2} \left[\ln \frac{C(s)}{R(s)} \right] = \frac{\frac{d^2}{ds^2} \left[\frac{C(s)}{R(s)} \right]}{\frac{C(s)}{R(s)}} - \left\{ \frac{\frac{d}{ds} \left[\frac{C(s)}{R(s)} \right]}{\frac{C(s)}{R(s)}} \right\}^2$$

– recall

$$\frac{C(s)}{R(s)} = 1 - \frac{E(s)}{R(s)} = 1 - \left(\frac{1}{1 + K_p} + \frac{1}{K_v} s + \frac{1}{K_a} s^2 + \dots \right)$$

Static Accuracy

– Therefore

$$\left\{ \frac{d^2}{ds^2} \left[\ln \frac{C(s)}{R(s)} \right] \right\}_{s=0} = -\frac{1}{K_v^2} - \frac{2}{K_a}$$

– Recall

$$-\left[\frac{d}{ds} \ln \frac{C(s)}{R(s)} \right]_{s=0} = \frac{1}{K_v} = \sum_{j=1}^n \frac{1}{p_j} - \sum_{j=1}^m \frac{1}{z_j}$$

Static Accuracy

– Therefore

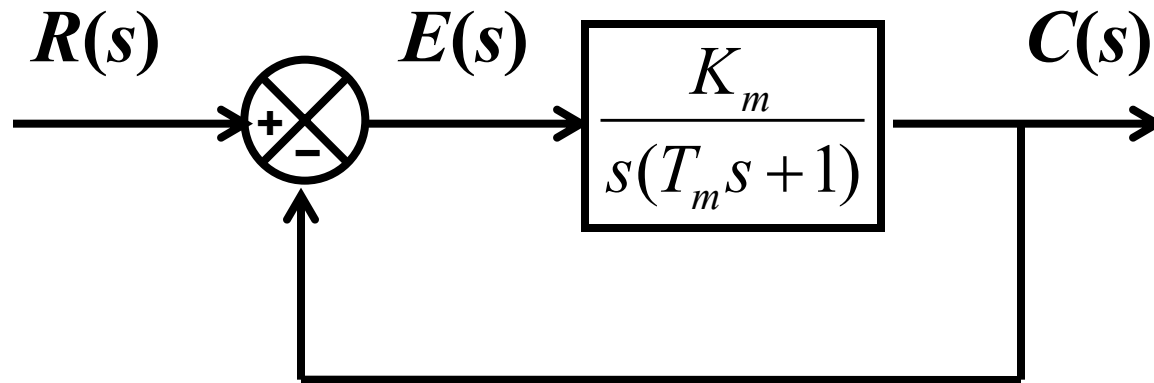
$$\sum_{j=1}^n \frac{1}{p_j^2} - \sum_{j=1}^m \frac{1}{z_j^2} = -\frac{1}{K_v^2} - \frac{2}{K_a}$$

- **Example**

$$\frac{C(s)}{R(s)} = \frac{K_m / T_m}{s^2 + (1/T_m)s + K_m / T_m}$$

Where K_m is the system gain and T_m is the time constant of the open-loop transfer function

Static Accuracy



- Or in the more familiar form

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Static Accuracy

- From the definition,

$$K_v = \lim_{s \rightarrow 0} sG(s) = \lim_{s \rightarrow 0} s \frac{K_m}{s(T_m s + 1)} = K_m$$

- therefore,

$$K_v = K_m = \frac{\omega_n}{2\zeta}$$

This is of great importance, to have very accurate response to velocity response, ζ must be very small—in inertial navigation applications, such as missile control.

Zero-Error System

- **General form,**

$$\begin{aligned}\frac{C(s)}{R(s)} &= \frac{G(s)}{1 + G(s)H(s)} \\ &= \frac{A_1 s^n + A_2 s^{n-1} + \cdots + A_n s + A_{n+1}}{B_1 s^m + B_2 s^{m-1} + \cdots + B_m s + B_{m+1}}\end{aligned}$$

- **Recall**

$$\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)} = \frac{1}{1 + K_p} + \frac{1}{K_v} s + \frac{1}{K_a} s^2 + \cdots$$

Zero-Error System

- Therefore

$$e_{ss} = \frac{r(t)}{1 + K_p} + \frac{\dot{r}(t)}{K_v} + \frac{\ddot{r}(t)}{K_a} + \dots$$

- It has been proved that for $e_{ss} = 0$, with a step input,

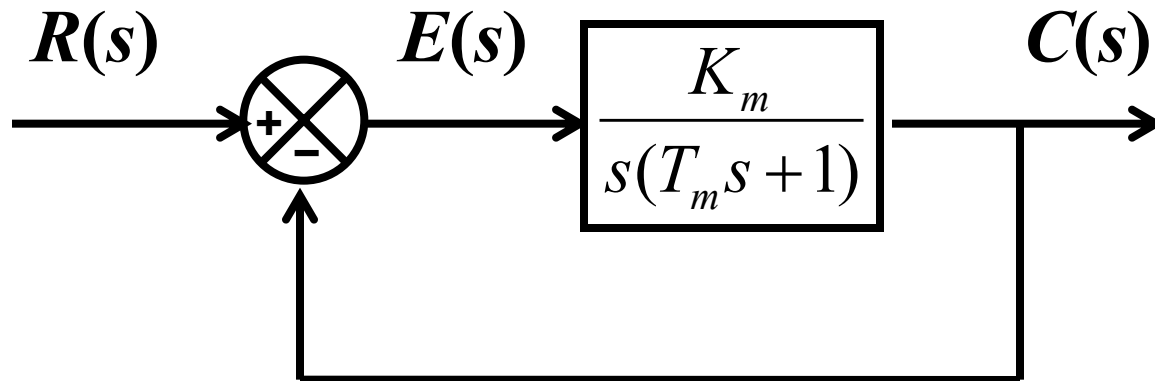
$$\frac{1}{1 + K_p} \text{ is a function of } \mathbf{B}_{m+1} - \mathbf{A}_{n+1}, \text{ and}$$
$$\mathbf{B}_{m+1} = \mathbf{A}_{n+1}$$

Zero-Error System

- **If,**
$$\frac{C(s)}{R(s)} = \frac{B_{m+1}}{B_1 s^m + B_2 s^{m-1} + \dots + B_m s + B_{m+1}}$$

The zero steady-state step error system

- **Example (one pure integrator)**



Zero-Error System

- **Example**

$$\begin{aligned}\frac{C(s)}{R(s)} &= \frac{G(s)}{1 + G(s)} \\ &= \frac{\frac{K_m}{s(T_m s + 1)}}{1 + \frac{K_m}{s(T_m s + 1)}} = \frac{K_m}{T_m s^2 + s + K_m}\end{aligned}$$

A zero steady-state step error system (prove it)

Zero-Error System

- It has also been proved that for $e_{ss} = 0$, with a ramp input,

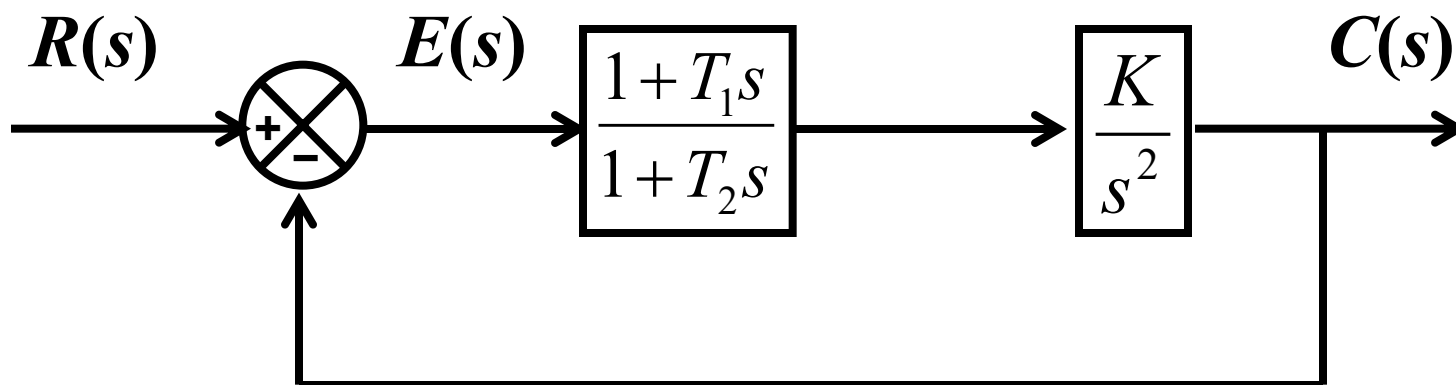
$$\frac{1}{K_v} \quad \text{is a function of } B_{m+1} - A_{n+1}, B_m - A_n, \text{ and } B_{m+1} = A_{n+1} \text{ and } B_m = A_n,$$

Zero-Error System

- It can be shown that for ramp input,

$$\frac{C(s)}{R(s)} = \frac{B_m s + B_{m+1}}{B_1 s^m + B_2 s^{m-1} + \dots + B_m s + B_{m+1}}$$

is a zero steady-state ramp error system



Zero-Error System

- It contains two pure integrates

$$\frac{C(s)}{R(s)} = \frac{\frac{T_1s + 1}{T_2s + 1} \frac{K}{s^2}}{1 + \frac{T_1s + 1}{T_2s + 1} \frac{K}{s^2}} = \frac{K(T_1s + 1)}{T_2s^2 + s^2 + KT_1s + K}$$

- It is a zero steady-state ramp error system